Problem 1

If we remove two edges from $K_6$, there are two cases. First, if the two edges have a vertex in common, then the subgraph on the other five vertices is $K_5$. This is clear because we removed no edges between any of these five vertices. Otherwise, if the two edges edges do not have a vertex in common, then $K_{3,3}$ is a subgraph of the result. For each removed edge, the vertices of that edge must be in the same component of the partition. This forces the removed edges to belong to different components of of the partition, so we see we have removed no edges between the two components.

In fact, $K_{3,3}$ is a subgraph in both cases, but for the first case, it is easier to see that $K_5$ is a subgraph. $K_5$ is not a subgraph of the second case.

Problem 2

For any $k \geq 0$ and $l \geq 0$, $r(k, l)$ is the smallest $n$ for which any edge two-coloring of $K_n$ contains either a blue $K_k$ or a green $K_l$. If $m < r(k, l)$, then there exists an edge two-coloring of $K_m$ that contains neither a blue $K_k$ nor a green $K_l$. If we reverse the colors in this coloring, then we have an edge two-coloring of $K_m$ that contains neither a blue $K_l$ nor a green $K_k$. This means $m < r(l, k)$. By taking $m = r(k, l) - 1$, we see that $r(k, l) \leq r(l, k)$.

Now if we consider a particular $r(k, l)$, we can apply this rule twice to get $r(k, l) \leq r(l, k) \leq r(k, l)$. Thus $r(k, l) = r(l, k)$.

Problem 3

We claim that $r(k, l, m) \leq r(k, r(l, m))$. To prove this, take an arbitrary edge 3-coloring of $K_{r(k, r(l, m))}$ (with colors blue, green, and red). Consider the corresponding 2-coloring where all the red edges are green. By the definition of $r(k, r(l, m))$, there is either a blue $K_k$ or a green $K_{r(l, m)}$. In the original 3-coloring, this corresponds to the existence of either a blue $K_k$ or a green and red $K_{r(l, m)}$. If there is a blue $K_k$, then we are done. Otherwise, we apply the definition of $r(l, m)$ to see that in the green and red $K_{r(l, m)}$ there is either a green $K_l$ or a red $K_m$.

Thus $r(k, l, m) \leq r(k, r(l, m))$. We know that these two argument Ramsey numbers exist and are finite, so the three argument Ramsey number exists and is finite.
Problem 4

Most methods of doing Problem 3 give a method for proving an explicit bound on any $r(k, l, m)$. We will proceed by using the inequality from Problem 3.

$$r(3, 3, 3) \leq r(3, r(3, 3)) = r(3, 6) = 18$$

Problems 5 and 6

We can prove that $r(3, 100) > 200$ by drawing an edge 2-coloring of $K_{200}$ with no blue triangle or green $K_{100}$. Thus Problems 5 and 6 are both generalizations of the following: Draw (or describe) an edge 2-coloring of $K_{2n}$ with no blue triangle or green $K_n$. (This will prove that $r(3, n) > 2n$.)

It is convenient to reformulate this problem in terms of cliques and independent sets. For $n \geq 4$, we will describe a graph on $2n$ vertices with no 3-independent set and no $n$-clique.

Call the vertices $w_1, w_2, v_1, \ldots, v_{n-1}, u_1, \ldots, u_{n-1}$.

Connect each $v_i$ to each $v_j$, and connect each $u_i$ to each $u_j$. Then connect $u_1$ to $v_1$ and $u_2$ to $v_2$. All the other edges will contain either $w_1$ or $w_2$.

Connect $w_1$ to every other vertex except $w_2, v_1,$ and $u_1$. Connect $w_2$ to every other vertex except $w_1, v_2,$ and $u_2$.

Consider all possible candidates for 3-independent sets. If $w_1$ is in the set, then the other vertices must be among $w_2, v_1,$ and $u_1$. But these three vertices are all mutually connected, so no 3-independent set contains $w_1$. Similarly, no 3-independent set contains $w_2$. Any set of three vertices that contains neither $w_1$ nor $w_2$ must contain either two $v_i$'s or two $u_j$'s. Therefore the graph has no 3-independent set.

Consider all possible candidates for $n$-cliques. $w_1$ and $w_2$ are not connected, so no clique contains both of them. Therefore at least $n - 1$ vertices of any clique are $v_i$'s and $u_j$'s. If the clique contains no $u_j$'s then it must contain all $v_i$'s and also either $w_1$ or $w_2$. But then the clique would be missing $(w_1, v_1)$ or $(w_2, v_2)$, so no such clique exists. Similarly, no clique exists that contains no $v_i$'s. But $n - 1 \geq 3$, so there are at least two $v_i$'s or at least two $u_j$'s. But there are no two $v_i$'s that connect to the same $v_j$, and there are no two $u_j$'s that connect to the same $v_i$. Thus no $n$-clique exists.

We exhibit this graph for $n = 5$ on the final page.
Problem 8

(a) Complete graphs

$K_n$ can trivially be $n$-colored by giving each vertex a different color. By the pigeonhole principle, any coloring of $K_n$ with fewer colors would have two vertices with the same color. But, in $K_n$, any two vertices are joined by an edge; in particular those vertices of the same color would be joined by an edge. Thus, the chromatic number of $K_n$ is $n$.

(b) Bipartite graphs

The vertices of any bipartite graph can be partitioned into two sets $V$ and $U$ such that there is no edge between any two vertices in $V$ and there is no edge between any two vertices of $U$. Any bipartite graph can then be 2-colored by picking a color for all the vertices of $V$ and picking another color for all the vertices in $U$. In fact, it is clear from this characterization that bipartite graphs are precisely those graphs which can be 2-colored.

Thus, the chromatic number of any bipartite graph is 2 unless the graph contains no edges, in which case its chromatic number is trivial.

(c) Trees

Consider the following naïve method for 2-coloring a connected 2-colorable graph. Start by picking a vertex. Color it red. Then, while there are still uncolored vertices, pick a vertex adjacent to a colored vertex. If the vertex is adjacent to any red vertices, color it blue, otherwise color it red.

We see that at every step of the algorithm, (except the initial choice of red which is chosen without loss of generality) we are forced to choose the color picked by the algorithm. Thus, if the algorithm fails (to produce a valid 2-coloring), then the input was not 2-colorable. If the algorithm succeeds, then we have exhibited a 2-coloring. Additionally, any 2-coloring is unique up to the names of the colors.

If this algorithm fails, then there is some step with an uncolored vertex $w$ adjacent to both a red vertex $u$ and a blue vertex $v$. This implies there is a colored path from $u$ to $v$. We also have an path from $u$ to $v$ through $w$. These paths are distinct, so we have a cycle.

Trees have no cycles, so the algorithm does not fail on trees.
Thus, the chromatic number of any tree is 2, unless the graph contains no edges, in which case its chromatic number is trivial.

(d) Polygons

Given a coloring of a graph, there is corresponding coloring of any subgraph. If the subgraph has as many vertices as the graph, then the colorings will have the same number of colors.

So, for a polygon of length $n$, consider the subgraph created by removing one edge. (By symmetry, it doesn’t matter which edge we remove.) The result is a tree, so by (c) it has a unique 2-coloring (unique up to the names of the colors). Therefore, if the original polygon had a 2-coloring, that coloring must correspond to the unique 2-coloring on the subgraph. Thus, the 2-coloring on the polygon can be recovered simply by using the 2-coloring from the subgraph.

For even $n$, we easily see that in any 2-coloring of the subgraph, the two vertices of degree one will have different colors, so we have a valid 2-coloring of the polygon. Furthermore, the polygon had an edge, so its chromatic number is 2.

For odd $n$, we just as easily see that in any 2-coloring of the subgraph, the two vertices of degree one will have the same color, so there is no valid 2-coloring of the polygon. To find a 3-coloring, now consider the subraph created by removing a vertex. The result is a tree, so it can be 2-colored. We extend this to a 3-coloring of the polygon by giving the removed vertex a third color. Thus, the chromatic number is 3.

(e) The Petersen graph

The Petersen graph contains an odd cycle. In other words, it contains an odd-length polygon as a subgraph, so any valid coloring of the Petersen graph must restrict to a valid coloring of the polygon. But any coloring of the polygon needs 3 colors, so any coloring of the Petersen graph must use at least 3 colors. We exhibit a 3-coloring on the following page.

(f)

The given graph contains an odd cycle (length 5), so as in (e), its chromatic number is at least 3. We exhibit a 3-coloring on the following page.
Problem 5:

Problem 8:

(e) with colors A, B, and C

(f)