Please note: You will be graded on both correctness and quality of exposition. Please show your work and explain your reasoning clearly.

“It is the eye of ignorance that assigns a fixed and unchangeable color to every object; beware of this stumbling block.” –Paul Gauguin

1. Show that if every pair of odd cycles of $G$ have a vertex in common, then $\chi \leq 5$.

2. Show that the only 3-critical graphs are the odd $k$-cycles with $k \geq 3$.

3. The join $G_1 \vee G_2$ of two graphs $G_1$ and $G_2$ is defined by connecting each vertex of $G_1$ to $G_2$. Show that $\chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2)$, and that $G_1 \vee G_2$ is critical if and only if both $G_1$ and $G_2$ are.

4. Assume $k \geq 3$. Let $G_1$ and $G_2$ be two $k$-critical graphs with exactly one vertex $v$ in common, and let $vv_1$ and $vv_2$ be edges of $G_1$ and $G_2$. Show that the graph $(G_1 - vv_1) \cup (G_2 - vv_2) + v_1v_2$ is $k$-critical.

5. Exhibit 4-critical graphs with $n$ vertices, for $n = 4$ and any $n \geq 6$. Moreover, prove there is no such graph with 5 vertices.

6. Calculate the chromatic polynomials of the graphs in Ex. 8.4.1, p. 128 of Bondy-Murty.

7. Calculate the chromatic polynomial of a polygon on $n$ vertices.

8. Beginning from Theorem 8.6 of Bondy-Murty (as was proved in class), prove that $\pi_k(G)$ is a polynomial in $G$ with leading term $k\#V(G)$. Moreover, if $G$ is simple, prove the second term is $-\#E(G)k^{\#V(G)-1}$.

9. Prove that if $G$ is connected on $n$ vertices, then $\pi_k(G) \leq k(k-1)^{n-1}$. (Hint: If you think about this, you can prove it with virtually no work.)

10. (Postponed to HW4.) Let $G$ be a simple graph with 10 vertices and 26 edges. Show that $G$ has at least 5 triangles. Can equality occur?

11. Extra credit: Without resorting to a long, tedious computation, prove that the Grötzsch graph cannot be 3-colored.