Homework 1 - Math 108, Frank Thorne (fthorne [at] math.stanford.edu)

Due Friday, October 1, at 4:30 p.m.

Please note: You will be graded on both correctness and quality of exposition. Please show your work and explain your reasoning clearly.

1. Prove that the following conditions are equivalent for a finite graph $G$:
   (1) $G$ is a tree (i.e. it is connected and has no cycles)
   (2) $G$ is connected, but removing any edge results in a disconnected graph.
   (3) $G$ has no cycles, but adding any edge creates a cycle.
   (4) $G$ is connected, and $\#V(G) - \#E(G) = 1$.

2. Prove that a finite simple ("simple" means no loops, and no two distinct edges have the same pair of ends) graph with at least two vertices must have at least two vertices of the same degree.

3. The girth of a graph is the length of the smallest polygon in the graph. Let $G$ be a graph of girth 5 for which all vertices have degree $\geq d$. Show that $G$ has at least $d^2 + 1$ vertices. Can equality hold?

4. Find the six nonisomorphic trees on 6 vertices, and for each compute the number of distinct labeled trees isomorphic to it. (See p. 13 of the book.)

5. Using Prüfer codes (rather than the second proof presented in class), count the number of labeled trees on 7 vertices where vertices 2 and 3 have degree 3, vertex 5 has degree 2, and all others have degree 1. (Count the number of possible Prüfer codes of such trees.)

6. Finish the Prüfer code proof that there are $n^{n-2}$ labeled trees on $n$ vertices. (In class, we described maps from trees to Prüfer codes and from codes to trees; you are asked to prove that these are inverses, and that these maps give a bijection.)

7. Suppose a tree $G$ has exactly one vertex of degree $i$ for each $2 \leq i \leq m$ and all other vertices have degree 1. How many vertices does $G$ have?

8. When we proved $K_5$ was nonplanar (in either lecture or Bondy-Murty) we assumed $v_4$ was on the interior of $C$. Prove the case where $v_4$ is on the exterior of $C$.

9. Prove directly (i.e. not using Euler’s formula) that the complete bipartite graph $K_{3,3}$ is nonplanar.

10. Prove that if you take one edge away from either $K_5$ or $K_{3,3}$, you get a planar graph.