1. Write the Hamilton equations for the system defined by the Lagrangian function

\[ L(q, \dot{q}) = (1 + q_1^2)\dot{q}_2^2 + (1 + q_2^2)\dot{q}_1^2, \quad q = (q_1, q_2) \in \mathbb{R}^2. \]

2. Let us recall that given a local coordinate system \((q_1, \ldots, q_n)\) on a manifold \(M\) one gets, in a canonical way, coordinates \((q_1, \ldots, q_n, p_1, \ldots, p_n)\) in the cotangent bundle \(T^*(M)\). The coordinate functions \(p_1, \ldots, p_n\) restricted to any cotangent space \(T^*_q(M)\) are just linear coordinates with respect to the basis of \(T^*_q(M)\) formed by the differentials \(dq_1, \ldots, dq_n\) computed at the point \(q\). We call the coordinates \(p_1, \ldots, p_n\) dual to \(q_1, \ldots, q_n\).

a) Express the coordinates \((r, \varphi, \rho, \theta)\) in \(T^*(\mathbb{R}^2)\) through the coordinates \((x_1, x_2, y_1, y_2)\).

b) Verify directly that if \((r, \varphi, \rho, \theta) = F(x_1, x_2, y_1, y_2)\). Then

\[ F^*(pdr + \theta d\varphi) = y_1 dx_1 + y_2 dx_2. \]

3. Having in mind the definition of the Hamiltonian vector field

\[ X_H : \mathcal{A} \to \mathcal{A}, \quad X_H \cdot d(pdq) = dH, \]

find the canonical Hamilton equations in coordinates \((r, \varphi, \rho, \theta)\) introduced above in Problem 2.

4. Suppose that the coordinates \((p, q)\) and \((P, Q)\) in \(\mathbb{R}^2\) are related by the formulas

\[ p = \frac{\partial f}{\partial q}(P, q), \]
\[ Q = \frac{\partial f}{\partial P}(P, q) \]

for some smooth function \(f(P, q)\). In other words, one gets identities by plugging in the above formulas the expressions \(P = P(p, q), Q = Q(p, q)\) of coordinates \((P, Q)\) through coordinates \((p, q)\). Show that the Hamiltonian systems

\[ \dot{P} = -\frac{\partial H}{\partial Q}(P, Q) \]
\[ \dot{Q} = \frac{\partial H}{\partial P}(P, Q) \]

and

\[ \dot{p} = -\frac{\partial \tilde{H}}{\partial q}(p, q) \]
\[ \dot{q} = \frac{\partial \tilde{H}}{\partial p}(p, q), \]

where \(\tilde{H}(p, q) = H(P(p, q), Q(p, q))\), are equivalent.

All problems including subproblems are 10 points each.