Math 52H Homework 6 Solutions

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February 20, 2011

1. Let $C_0 = [0, 1]$, and for $n \geq 1$, let $C_n$ denote the set obtained from $C_{n-1}$ by removing the central open subinterval of every interval in $C_{n-1}$. We have that $C = \cap_n C_n$, and the volume of $C_n$ is $(2/3)^n$. Thus $C$ is covered by Riemann measurable sets $C_n$ of arbitrarily small volume, hence $C$ is measurable with volume 0 by Prop 4.4.4.

2.a. Write $\alpha = \frac{xdx + ydy + zdz}{(x^2 + y^2 + z^2)^{3/2}}$ and let $G : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ be defined by $G(x, y, z) = -(x^2 + y^2 + z^2)^{3/2}$. Then

$$dG = -(x^2 + y^2 + z^2)^{3/2} (-\frac{1}{2})(2xdx + 2ydy + 2zdz) = \alpha$$

so $\alpha$ is exact on $\mathbb{R}^3 \setminus \{0\}$, which contains the curve $\gamma$. It follows that

$$\int_\gamma \alpha = G(\gamma(1)) - G(\gamma(0)) = \left(\frac{1}{\sqrt{6}} - \frac{1}{2\sqrt{3}}\right).$$

b. Write $\alpha = \frac{ydx - ydx}{x^2 + y^2}$. On the set $\{(x, y) : x > 0\}$ we have $\alpha = d\arctan(y/x)$, so $\gamma$ is exact on this domain. Since $\gamma_x(t) = 1 + \frac{1}{2} \sin \frac{t^2}{\pi} > 0$ for all $t$, the curve $\gamma$ is contained in $\{(x, y) : x > 0\}$. Hence

$$\int_\gamma \alpha = \arctan(\frac{\gamma_y(pi)}{\gamma_x(pi)}) - \arctan(\frac{\gamma_y(0)}{\gamma_x(0)}) = 0 - \frac{\pi}{4} = -\frac{\pi}{4}.$$

3. a. Write $\alpha = \alpha_1dx + \alpha_2dy + \alpha_3dz$. Assume that there is a primitive $F$. If we integrate $\alpha_1$ with respect to $x$, we get

$$F = x^3/3 - x^2/2 + xy + xz - xyz + f(y, z),$$
for some function \( f(y, z) \) depending only on \( y \) and \( z \). Differentiate this w.r.t. \( y \) and set it equal to \( \alpha_2 \) to get

\[
x - xz + \frac{\partial f}{\partial y} = y^2 + x - y + z - xz,
\]

which is equivalent to \( f = y^3/3 - y^2/2 + yz + g(z) \) for some function \( g \) depending only on \( z \). Now differentiate the expression we have for \( F \) w.r.t. \( z \) and set equal to \( \alpha_3 \) to get

\[
x - xy + y + g'(z) = z^2 + x + y - z - xy,
\]

whence \( g(z) = z^3/3 - z^2/2 + C \) for some constant \( C \). We thus have

\[
F(x, y, z) = \frac{x^3 + y^3 + z^3}{3} - \frac{x^2 + y^2 + z^2}{2} + xy + xz + yz - xyz + C.
\]

0.1 Remark:

It is important to note that the fact that we were able to find a solution to the equations above implies that \( \alpha \) is exact with \( F \) as its primitive. In fact, the equations we solved above imply that \( dF = \alpha \). Also, note that finding a primitive for an exact 1-form is the same as finding a potential function for a vector field. Finally, we remark here that one can guess the form of \( F \) very easily by symmetry.

b. Since \( \alpha = dF \), we have that

\[
\int_\gamma \alpha = F(\gamma(\pi)) - F(\gamma(0)) = 0,
\]

since \( \gamma(\pi) = \gamma(0) \). In other words, \( \gamma \) is a closed path, and so the integral of the exact form \( \alpha \) along \( \gamma \) is 0.