1. Let $e_1, ..., e_n, f_1, ..., f_n$ be the standard basis on $\mathbb{R}^{2n}$ with $x_1, ..., x_n, y_1, ..., y_n$ the dual basis. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be given by $Ae_i = \sum_{j=1}^n A_{ij} e_j$.

   a. We have
   \[
   (L_A)^* \omega = \sum_{1 \leq i < j \leq n} (L_A)^* \omega(e_i, e_j) x_i \wedge x_j.
   \]
   Now
   \[
   (L_A)^* \omega(e_i, e_j) = \omega(L_A e_i, L_A e_j)
   \]
   \[
   = \sum_{l=1}^n (x_l \otimes y_l - y_l \otimes x_l)((e_i + \sum_{k=1}^n A_{ki} f_k), (e_j + \sum_{k=1}^n A_{kj} f_k))
   \]
   \[
   = \sum_{l=1}^n \left[ x_l(e_i + \sum_{k=1}^n A_{ki} f_k)y_l(e_j + \sum_{k=1}^n A_{kj} f_k) - y_l(e_i + \sum_{k=1}^n A_{ki} f_k)x_l(e_j + \sum_{k=1}^n A_{kj} f_k) \right]
   \]
   \[
   = y_i(e_j + \sum_{k=1}^n A_{kj} f_k) - y_j(e_i + \sum_{k=1}^n A_{ki} f_k)
   \]
   \[
   = A_{ij} - A_{ji}.
   \]
   So $(L_A)^* \omega = \sum_{1 \leq i < j \leq n} (A_{ij} - A_{ji}) x_i \wedge x_j$ and $(L_A)^* \omega = 0$ if and only if $A$ is symmetric.

   b. Write $C_\omega(e_k) = \sum_j (\alpha_j x_j + \beta_j y_j)$. We have that $\alpha_j = C_\omega(e_k)(e_j) = \omega(e_k, e_j) = \sum_i x_i \wedge y_i(e_k, e_j) = 0$, and $\beta_j = C_\omega(e_k)(f_j) = \omega(e_k, f_j) = \sum_i x_i \wedge y_i(e_k, f_j) = \delta_{k,j}$.

   Similarly, write $C_\omega(f_k) = \sum_j (\alpha_j x_j + \beta_j y_j)$. We then have $\alpha_j = C_\omega(f_k)(e_j) = \omega(f_k, e_j) = \sum_i x_i \wedge y_i(f_k, e_j) = -\delta_{k,j}$, and $\beta_j = 0$ similar to before.

   The vector $(\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n)$ corresponding to $(e_k, f_k)$ is the $k$th column of the required matrix. Thus $C = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, where $0$ denotes the $n \times n$ zero matrix, and $I$ denotes the $n \times n$ identity matrix.

2. Consider the linear operator $L = \frac{1}{2}(* + I) : \Lambda^2((\mathbb{R}^4)^*) \to \Lambda^2((\mathbb{R}^4)^*)$; the two-form $\beta$ is self-dual if and only if $L \beta = \beta$. 
Since $s^2 = I$ is the identity, $L^2 = \frac{1}{4}(s^2 + 2s + I) = \frac{1}{2}(s + I) = L$ so $L$ is a projection, that is,

$$\{v : Lv = v\} = \text{im}(L).$$

We have

$$L(x_1 \wedge x_2) = \frac{1}{2}(x_1 \wedge x_2 + x_3 \wedge x_4) = L(x_3 \wedge x_4), \quad L(x_1 \wedge x_3) = \frac{1}{2}(x_1 \wedge x_3 - x_2 \wedge x_4) = -L(x_2 \wedge x_4)$$

$$L(x_1 \wedge x_4) = \frac{1}{2}(x_1 \wedge x_4 + x_2 \wedge x_3) = L(x_2 \wedge x_3),$$

so the space of self-dual forms is 3-dimensional, with basis

$$\left\{ \frac{1}{2}(x_1 \wedge x_2 + x_3 \wedge x_4), \frac{1}{2}(x_1 \wedge x_3 - x_2 \wedge x_4), \frac{1}{2}(x_1 \wedge x_4 + x_2 \wedge x_3) \right\}.$$  

3. We let $A$ be the matrix representation of $A$, and let $e_1, ..., e_n$ be the basis of $\mathbb{R}^n$ corresponding to $x_1, ..., x_n$. Then apply the identity given in the question to $(e_1, e_2, ..., e_n)$. The left hand side is $\det A$. The right hand side is, by definition of exterior product

$$\sum_{i_1 < i_2 < ... < i_k, i_{k+1} < ... < i_n} (-1)^{\text{inv}(i_1, ..., i_n)} A^* (x_1 \wedge ... \wedge x_k)(e_{i_1}, ..., e_{i_k}) A^* (x_{k+1} \wedge ... \wedge x_n)(e_{i_{k+1}}, ..., e_{i_n}),$$

and this is precisely the right hand side by Proposition 1.9.5 in the notes.

4. Let $Y = \alpha X + \beta Z$ where $< X, Z > = 0$. Then $X \times Y = \beta X \times Z$, since $X \times X = 0$ whereas $D^{-1}(*(D(X) \wedge D(Y))) = \beta D^{-1}(*(D(X) \wedge D(Z)))$ since $D(X) \wedge D(X) = 0$. It thus suffices to check that $X \times Z = D^{-1}(*(D(X) \wedge D(Z)))$.

We may assume that $X, Z \neq 0$, and by dividing the above by $||X|| ||Z||$, we may further assume that $X$ and $Z$ are orthonormal. Let $X, Z, W$ be an orthonormal basis for $\mathbb{R}^3$, where $W = X \times Z$, so that the basis defines the standard orientation for $\mathbb{R}^3$. Note that $D(X), D(Z), D(W)$ forms the dual basis. By §1.16 in the notes \(^1\), we have that $*(D(X) \wedge D(Z)) = D(W)$, and so $D^{-1}(*(D(X) \wedge D(Z))) = W = X \times Z$ as desired.

5. We first prove the following lemma.

**Lemma 1.** For any two $n \times n$ matrices $A$ and $B$ such that $AB = BA$, we have that

$$\exp(A + B) = \exp(A) \exp(B).$$

\(^1\)See, for instance, the second example given in that section.
Proof.

\[
\exp(A + B) = \sum_{k \geq 0} \frac{(A + B)^k}{k!} \\
= \sum_{j \geq 0} \sum_{m=0}^{k} \frac{A^m B^{j-m}}{m!(j-m)!} \\
= \sum_{m,n} \frac{A^m B^n}{m! n!},
\]
as desired. In the above, we have used that \(AB = BA\) in our binomial expansion.

Next note that for any \(n \times n\) matrix \(M\) that \(\exp(M)^T = \left( \sum_{k \geq 0} \frac{M^k}{k!} \right)^T = \sum_{k \geq 0} \frac{(M^T)^k}{k!} = \exp(M^T)\). Now let \(A\) be skew-symmetric so \(A^T = -A\). Then \(A^T A = AA^T\) so applying the above and the Lemma gives

\[
\exp(A) \exp(A)^T = \exp(A) \exp(A^T) = \exp(A + A^T) = \exp(0) = I,
\]
so \(\exp(A)\) is orthogonal.

Conversely, say that \(\exp(tA)\) is orthogonal for all \(t\). Then

\[
I = \exp(tA) \exp(tA^T) \\
= \left( \sum_{k \geq 0} \frac{t^k A^k}{k!} \right) \left( \sum_{k \geq 0} \frac{t^k (A^T)^k}{k!} \right) \\
= I + t(A + A^T) + \frac{t^2}{2} (2AA^T + A^2 + (A^T)^2) + ...
\]

Apply \(\frac{d}{dt} \bigg|_{t=0}\) to the equation above. The left hand side is 0, whereas the right hand side is \((A + A^T + 2t(2AA^T + A^2 + (A^T)^2) + ...)\bigg|_{t=0} = A + A^T\). Hence \(A = -A^T\), as desired.