1. Consider the space $\mathbb{R}^{2n}$ with coordinates $(x_1, y_1, \ldots, x_n, y_n)$. Denote $\omega = \sum_{1}^{n} x_i \wedge y_i$.

A linear operator $A : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is called symplectic if $A^* \omega = \omega$.

Denote

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & -1 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & -1 \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$ 

Note that $J^2 = -E$. Let $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the operator with the matrix $J$.

An operator $U : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is called unitary if it is orthogonal and commutes with $J$, i.e. $U \circ J = J \circ U$. Respectively, a matrix $U$ is called unitary, if it is orthogonal and $UJ = JU$.

(a) Let $A$ be symplectic. Prove that $\det A = 1$.

(b) Prove that an operator $A$ is symplectic if and only if its matrix $A$ (in the standard basis of $\mathbb{R}^{2n}$) satisfies the equation $A^TJA = J$.

(c) Prove that an orthogonal operator $U : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is unitary if and only if it is symplectic.
(d) An \(n\)-dimensional vector subspace \(L \subset \mathbb{R}^{2n}\) is called Lagrangian if \(\omega|_L = 0\). Prove that \(L\) is Lagrangian if and only if \(\mathcal{J}(L) = L^\perp\).

(e)* Denote \(L_0 = \{y_1 = y_2 = \cdots = y_n = 0\}\). Prove that a subspace \(L \subset \mathbb{R}^{2n}\) is Lagrangian if and only if there exists a unitary operator \(U\) such that \(L = U(L_0)\).

2. In \(\mathbb{R}^2\) with the standard dot-product consider a basis \(v_1 = (1, 0), v_2 = (1, 1)\). Let \((y_1, y_2)\) be coordinates dual to this basis. Given a function \(f(y_1, y_2)\) compute its gradient in these coordinates, i.e. find functions \(g_1(y_1, y_2), g_2(y_1, y_2)\) such that \(\nabla f = g_1 \frac{\partial}{\partial y_1} + g_2 \frac{\partial}{\partial y_2}\).

3. Let \(X = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}, Y = \sum_{i=1}^n g_i \frac{\partial}{\partial x_i}\) be two vector fields on \(\mathbb{R}^n\). Prove that there exists a vector field \(Z = \sum_{i=1}^n h_i \frac{\partial}{\partial x_i}\) such that \(D_X D_Y - D_Y D_X = D_Z\), and find an explicit formula for the coordinate functions \(h_i\) of \(Z\) in terms of coordinate functions of the vector fields \(X, Y\).

All problems and subproblems are 10 points. Problem 1(e) is not mandatory.