PRACTICE MIDTERM

(1) Time: 1.5 hours
(2) Answer ALL parts of ALL the following questions. Give full proofs for any claim you make.
(3) This is a closed books and notebooks exam.

I (a) Prove that $l^\infty$, the space of all bounded sequences, is a Banach space with respect to the norm $\|x\| := \sup_{n=1,2,\ldots} |x_n|$.
(b) Is it a Hilbert space? Why?

II Prove the following inequality between the mean and the mean square:
$$\frac{1}{n} \left| \sum_{i=1}^{n} x_i \right| \leq \left\{ \frac{1}{n} \sum_{i=1}^{n} |x_i|^2 \right\}^{1/2}$$

III Show that a compact metric space is always a complete metric space.
Reminder: Space is compact if any sequence of vectors of the space has a convergent subsequence.

IV Consider the space of continuous complex valued functions on the unit interval $C[0, 1]$ as an inner product space with the inner product
$$(f, g) := \int_{0}^{1} f(t)\overline{g(t)}dt .$$

Define $e_0(t) = 1$, $e_1(t) = \sqrt{3}(2t - 1)$ (remember that $0 \leq t \leq 1$).
(a) Show that $e_0, e_1$ are an orthonormal system.
(b) Find the polynomial $y$ of the degree one which is closest with respect to the norm $\|f\|^2 = (f, f)$ to the function $x(t) = t^2$ (where again $0 \leq t \leq 1$).