HOMEWORK ASSIGNMENT 6

(1) Solve problems 7.2, 7.4, 7.8, 7.12

(2) Read the handout about weak convergence.

(3) Prove that if, in a separable Hilbert space, \( x_n \to x \) weakly and \( \|x_n\| \to \|x\| \), then \( \|x_n\| \to \|x\| \) in the norm sense. (Hint: Expand \( (x - x_n, x - x_n) \), and consider the limit of each term as \( n \to \infty \).)

(4) (a) Prove that an operator \( T \in \mathcal{L}(H) \) (\( H \) is a Hilbert space) that is \( 1-1 \), has dense range, and is bounded below (i.e., \( \inf_{\|x\|=1} \|Tx\| > 0 \)) is invertible. (Hint: Note that you were not given that \( T \) is onto, i.e. that for every \( x \) there is a \( y \) so that \( x = Ty \). Proving this will be your first step toward establishing that \( T \) is invertible. First show that if \( y_i \in H \) and \( T \) is bounded below then if \( Ty_i \to x \) then \( y_i \) is a Cauchy sequence.

(b) Concluded that if \( T \) fails to be invertible, it fails for one of the following mutually exclusive reasons:

S 1. \( T \) is not \( 1-1 \), i.e., it has a non trivial kernel: \( \ker T = \{ x : Tx = 0 \} \neq 0 \).
S 2. \( T \) is \( 1-1 \) and \( T(H) \) is dense in \( H \), but \( T \) is not bounded below (see above).
S 3. \( T \) is \( 1-1 \), but \( T(H) \) is not dense in \( H \).
   Give an example for each case.