Math 113 Midterm Solutions
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Question 1. Let $V$ and $W$ be finite dimensional vector spaces and $T \in \mathcal{L}(V, W)$ a linear transformation.

(a) Show that there is a basis $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$ of $V$ so that for some $k$, \{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$ are linearly independent and $T(\vec{v}_{k+1}) = T(\vec{v}_{k+2}) = \ldots = T(\vec{v}_n) = 0$.

Proof. We first present a solution which uses subspace ideas; the second solution uses explicit bases. Let $U := \ker(T)$. Note that $U$ is a subspace of $V$; thus we may write $V$ as a direct sum, $V = U \oplus U'$, for some other subspace $U' \subset V$, by Theorem 3.21. Consider $T$ restricted to $U'$. This is a map $T : U' \to W$, which is injective, since $U' \cap \ker(T) = \{0\}$. And note that injective maps, in general, preserve linear independence: if we have a linearly independent set \{\vec{x}_1, \ldots, \vec{x}_n\} in a vector space $X$, a linear map of vector spaces $A : X \to Y$, and a linear combination $0 = \lambda_1 A\vec{x}_1 + \cdots + \lambda_n A\vec{x}_n$, then $0 = A(\lambda_1 \vec{x}_1 + \cdots + \lambda_n \vec{x}_n)$, so $\lambda_1 \vec{x}_1 + \cdots + \lambda_n \vec{x}_n = 0$, which implies $\lambda_1 = \cdots = \lambda_n = 0$. Hence, choosing arbitrary bases for $U$ and $U'$ automatically gives us the desired properties.

Alternatively: Since $\ker(T)$ is a subspace, we may choose a basis \{\vec{v}_n, \vec{v}_{n-1}, \ldots, \vec{v}_{k+1}\} for it. Then by Theorem 2.12, we can extend this to a basis \{\vec{v}_1, \ldots, \vec{v}_n\} of $V$. Since $\{\vec{v}_{k+1}, \ldots, \vec{v}_n\} \in \ker(T)$ we get by definition that $T(\vec{v}_{k+1}) = \cdots = T\vec{v}_n = 0$. We claim that in addition, $\{T\vec{v}_1, \ldots, T\vec{v}_k\}$ is a linearly independent set of vectors. Indeed, suppose that there exists a linear combination

$$\lambda_1 T\vec{v}_1 + \cdots + \lambda_k T\vec{v}_k = 0 = T(\lambda_1 \vec{v}_1 + \cdots + \lambda_k \vec{v}_k).$$

This implies that $\lambda \vec{v}_1 + \cdots + \lambda_k \vec{v}_k$ is in the kernel of $T$, and hence it can be expressed in the basis:

$$\lambda \vec{v}_1 + \cdots + \lambda_k \vec{v}_k = \nu_1 \vec{v}_{k+1} + \cdots + \nu_{n-k} \vec{v}_n.$$

However, since the \{\vec{v}_i\}_{i=1}^n form a basis, this implies that all the $\lambda_1 = \cdots = \lambda_k = 0$, as desired. \qed

(b) With $\vec{v}_i$ as above, show that there is a basis $(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m)$ of $W$ so that $T(\vec{v}_1) = \vec{w}_1, T(\vec{v}_2) = \vec{w}_2, \ldots, T(\vec{v}_k) = \vec{w}_k$.

Proof. This follows from the fact that any subspace of a vector space is a direct summand. In particular, since $\text{Im}(T) \subset W$, there exists $W' \subset W$ such that $\text{Im}(T) \oplus W' = W$. 

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and choosing bases for both direct summands then gives us the desired basis for $W$.

More explicitly, define $\vec{w}_1 = T\vec{v}_1$, $\vec{w}_2 = T\vec{v}_2$, ..., $\vec{w}_k = T\vec{v}_k$. From (a) we know that this set is linearly independent; hence it may be enlarged to a basis $\{\vec{w}_1, \ldots, \vec{w}_m\}$ for $W$, by Theorem 2.12, which is of course finite since $W$ is finite-dimensional.

(c) With $\vec{v}_i$ and $\vec{w}_i$ as above, what is the matrix $\mathcal{M}(T, (\vec{v}_1, \ldots, \vec{v}_n), (\vec{w}_1, \ldots, \vec{w}_m))$?

Justify your answer.

Proof. Note that we have, from (a) and (b),

- $T\vec{v}_1 = 1\vec{w}_1 + 0\vec{w}_2 + \cdots + 0\vec{w}_m$,
- $T\vec{v}_2 = \vec{w}_2$,
- $T\vec{v}_k = \vec{w}_k$,
- $T\vec{v}_{k+1} = 0$,
- $T\vec{v}_n = 0$.

Thus, the matrix must look like

$$
\begin{pmatrix}
1 & & & \\
& 1 & & \\
& & \ddots & \\
0 & & & 1
\end{pmatrix}
$$

where all off-diagonal entries are 0, and there are exactly $k$ 1’s on the diagonal.

Question 2. Given a vector space $V$ over $\mathbb{R}$, there is a standard way of making it into a vector space over $\mathbb{C}$. Define $V_\mathbb{C} = \{\vec{u} + \vec{v}i : \vec{u}, \vec{v} \in V\}$.

Define addition and scalar multiplication by:

- $(\vec{u} + \vec{v}i) + (\vec{u}' + \vec{v}'i) = (\vec{u} + \vec{u}') + (\vec{v} + \vec{v}')i$,
- $(a + bi) \cdot (\vec{u} + \vec{v}i) = (a \cdot \vec{u} - b \cdot \vec{v}) + (a \cdot \vec{v} + b \cdot \vec{u})i$.

(a) Show that $V_\mathbb{C}$ is a vector space over $\mathbb{C}$.

Proof. We verify the properties of a vector space.

Commutativity:

- $(\vec{u} + \vec{v}i) + (\vec{u}' + \vec{v}'i) = (\vec{u} + \vec{u}') + (\vec{v} + \vec{v}')i$,

using commutativity in $V$.

Associativity:

- $(\vec{u} + \vec{v}i) + ((\vec{u}' + \vec{v}'i) + (\vec{u}'' + \vec{v}'')i) = (\vec{u} + \vec{v}i) + ((\vec{u}' + \vec{v}') + (\vec{v} + \vec{v}'')i)$
- $= (\vec{u} + (\vec{u}' + \vec{v}'i)) + ((\vec{v} + (\vec{v} + \vec{v}'')i)i = (\vec{u} + \vec{v}') + (\vec{v}') + (\vec{v} + \vec{v}'')i$
- $= ((\vec{u} + \vec{u}') + (\vec{v} + \vec{v}')i) + (\vec{u}'' + \vec{v}'')i = ((\vec{u} + \vec{v}) + (\vec{v} + \vec{v}')i) + (\vec{u}'' + \vec{v}'')i$;
and
\[
((a + bi)(c + di))(\vec{u} + \vec{v}i) = (ac - bd + (ad + bc)i)(\vec{u} + \vec{v}i)
\]
\[
= (((ac - bd)\vec{u} - (ad + bc)\vec{v}) + ((ad + bc)\vec{u} + (ac - bd)i)\vec{v}) + ((ad + bc)\vec{u} + (ac - bd)i)\vec{v})
\]
\[
= (a(c\vec{u} - d\vec{v}) - b(d\vec{u} + c\vec{v}) + (a(d\vec{u} + c\vec{v}) + b(c\vec{u} - d\vec{v}))i
\]
\[
= (a + bi)((c\vec{u} - d\vec{v}) + (d\vec{u} + c\vec{v})i)
\]
\[
= (a + bi)((\vec{u} + \vec{v}i) + (c + di)(\vec{u} + \vec{v}i)).
\]

The additive identity is \(\vec{0} = (\vec{0} + \vec{0}i)\), since
\[
(\vec{0} + \vec{0}i) + (\vec{u} + \vec{v}i) = (\vec{u} + \vec{0}) + (\vec{v} + \vec{0})i = \vec{u} + \vec{v}i.
\]

Additive inverses exist: \(- (\vec{u} + \vec{v}i) = (- \vec{u}) + (- \vec{v})i\), since
\[
(\vec{u} + \vec{v}i) + ((- \vec{u}) + (- \vec{v})i) = (\vec{u} + - \vec{u}) + (\vec{v} + - \vec{v})i = \vec{0} + \vec{0}i = \vec{0}.
\]

The multiplicative identity is \(1 = 1 + 0i\):
\[
(1 + 0i)(\vec{u} + \vec{v}i) = (\vec{u} - 0\vec{v}) + (\vec{v} + 0\vec{u})i = \vec{u} + \vec{v}i.
\]

Finally, we verify the distributive properties:
\[
(a + bi)((\vec{u} + \vec{v}i) + (\vec{u}' + \vec{v}'i)) = (a + bi)((\vec{u} + \vec{u}') + (\vec{v} + \vec{v}')i)
\]
\[
= (a(a\vec{u} + \vec{u}') - b(\vec{v} + \vec{v}')) + (a(\vec{v} + \vec{v}') + b(\vec{u} + \vec{u}'))i
\]
\[
= (a\vec{u} - b\vec{v} + a\vec{u}' - b\vec{v}') + (a\vec{v} + b\vec{u} + a\vec{v}' + b\vec{u}')i
\]
\[
= ((a\vec{u} - b\vec{v}) + (a\vec{v} + b\vec{u}))i + ((a\vec{u}' - b\vec{v}') + (a\vec{v}' + b\vec{u}')i)
\]
\[
= (a + bi)((\vec{u} + \vec{v}i) + (\vec{u}' + \vec{v}'i));
\]

and
\[
((a + bi) + (c + di))(\vec{u} + \vec{v}i) = (a + c + (b + d)i)(\vec{u} + \vec{v}i)
\]
\[
= ((a + c)\vec{u} - (b + d)i\vec{v}) + ((a + c)i\vec{v} + (b + d)\vec{u})i
\]
\[
= (a\vec{u} - b\vec{v} + c\vec{u} - d\vec{v}) + (a\vec{v} + b\vec{u} + c\vec{v} + d\vec{u})i
\]
\[
= ((a\vec{u} - b\vec{v}) + (a\vec{v} + b\vec{u})i) + ((c\vec{u} - d\vec{v}) + (c\vec{v} + d\vec{u})i)
\]
\[
= (a + bi)(\vec{u} + \vec{v}i) + (c + di)(\vec{u} + \vec{v}i).
\]

(b) Suppose that \(V\) is finite dimensional. Show that the dimension of \(V\) as a real vector space equals the dimension of \(V_C\) as a complex vector space.

Proof. Let the real dimension of \(V\) be \(n\), so that there exists a basis \(\{\vec{v}_1, \ldots, \vec{v}_n\}\) of \(V\) as a real vector space.

We claim that \(\{\vec{v}_1 + \vec{0}_i, \ldots, \vec{v}_n + \vec{0}_i\}\) is a basis for \(V_C\) as a complex vector space.

Indeed, consider any arbitrary vector \(\vec{u} + \vec{w}i \in V_C\). Since \(\vec{u}, \vec{w} \in V\) are real vectors, we can write
\[
\vec{u} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n,
\]
\[
\vec{w} = b_1\vec{v}_1 + \cdots + b_n\vec{v}_n,
\]
where the \(a_i, b_i\) are real numbers. Hence, we get that
\[
\vec{u} + \vec{w}i = (a_1 + b_1i)\vec{v}_1 + \cdots + (a_n + b_ni)\vec{v}_n,
\]

\[
\square
\]
in other words, $\bar{u} + w\bar{i}$ can be written as a complex linear combination of $\{\bar{v}_1, \ldots, \bar{v}_n\}$. Since $\bar{u} + w\bar{i}$ was an arbitrary vector in $V_C$, this shows that $\{\bar{v}_1, \ldots, \bar{v}_n\}$ spans $V_C$.

Next, to show that the set is linearly independent in $V_C$, note that if we have a complex linear combination

$$0 + 0i = (a_1 + b_1 i)\bar{v}_1 + \cdots + (a_n + b_n i)\bar{v}_n,$$

then by equating real and imaginary parts,

$$0 = a_1 \bar{v}_1 + \cdots + a_n \bar{v}_n, \quad 0 = b_1 \bar{v}_1 + \cdots + b_n \bar{v}_n,$$

which implies that $a_1 = \cdots = a_n = b_1 = \cdots = b_n = 0$, since the $\{\bar{v}_1, \ldots, \bar{v}_n\}$ are linearly independent as real vectors. Since the only complex linear combination of the $\bar{v}_i$ that is 0 is the trivial one, this implies that $\{\bar{v}_1, \ldots, \bar{v}_n\}$ is linearly independent, so it is a basis for $V_C$.

Therefore the complex vector space dimension of $V_C$ is the same as the real vector space dimension of $V$, as desired. \hfill \Box

**Question 3.** Let $U, V$ and $W$ be finite dimension vector spaces of dimension at least one. Let $R \in \mathcal{L}(U, W)$ and $S \in \mathcal{L}(V, W)$ be linear transformations.

(a) Show that there exists a $T \in \mathcal{L}(U, V)$ so that $R = ST$ if and only if $\text{Im}(R) \subset \text{Im}(S)$.

**Proof.** We have the diagram:

$$\begin{array}{ccc}
U & \overset{R}{\longrightarrow} & W \\
\downarrow{T} & & \downarrow{S} \\
V & \overset{}{\longrightarrow} & \end{array}$$

Note that if $R = ST$, then $\text{Im}(R) = \text{Im}(ST) = \{Sv : v \in \text{Im}(T)\} \subset \text{Im}(S)$, as desired. Thus it remains to show the converse.

Assume that $\text{Im}(R) \subset \text{Im}(S)$. By applying the results of problem 1a and 1b to $S$, we obtain a basis $\{\bar{w}_1, \ldots, \bar{w}_p\}$ for $W$ and a basis $\{\bar{v}_1, \ldots, \bar{v}_m\}$ for $V$ such that

$$S\bar{v}_1 = \bar{w}_1, \ldots, S\bar{v}_k = \bar{w}_k, S\bar{v}_{k+1} = 0, \ldots, S\bar{v}_m = 0.$$

This also implies that $\{\bar{w}_1, \ldots, \bar{w}_k\}$ is a basis for $\text{Im}(S)$.

Now define the map $S' : W \rightarrow V$ by

$$S'\bar{w}_1 = \bar{v}_1, \ldots, S'\bar{w}_k = \bar{v}_k, S'\bar{w}_{k+1} = 0, \ldots, S'\bar{w}_p = 0,$$

and it is then easy to see that $SS'\bar{v}_i = \bar{w}_1, \ldots, SS'\bar{v}_k = \bar{w}_k$, i.e. $SS'$ is the identity on $\text{Im}(S) \subset W$.

Alternatively, to construct such a $S'$ more abstractly, without using the results of problem 1, note that $\ker(S) \subset V$ is a subspace. Thus, by Theorem 2.12 we may write

$$V = \ker(S) \oplus V',$$

for some other subspace $V' \subset V$.

Then $S$ is injective when restricted to $V'$ (since $\ker(S) \cap V' = \{0\}$), hence every $\bar{w} \in \text{Im}(S)$ has a unique pre-image $\bar{v} \in V'$. But $S$ is clearly surjective onto its image, hence we get a bijection between $V' \to \text{Im}(S)$.

Hence we can simply define $S'$ to be the inverse of this map, i.e. $S' \in \mathcal{L}(\text{Im}(S), V')$, and redefine its codomain to be the whole of $V$. This redefinition of codomain means that $SS'$ is no longer the identity, but $SS'$ remains an identity map from $\text{Im}(S) \to \text{Im}(S)$.

Now let $T := S'R$. Since $\text{Im}(R) \subset \text{Im}(S)$, and $S'$ was constructed so that $SS'$ is the identity on $\text{Im}(S)$, it follows that $ST = SS'R = R$, as desired. \hfill \Box
(b) Assume that such a \( T \) exists. Show that there is only one such \( T \) if and only if \( S \) is injective.

Proof. Assume \( S \) is injective, and suppose we have two maps \( T, T' \) such that

\[
R = ST, \quad R = ST'.
\]

Subtracting these two equations (which hold when applied to any \( \vec{u} \in U \)), we get

\[
ST\vec{u} - ST'\vec{u} = S(T - T')\vec{u} = 0.
\]

Since \( S \) is injective, the only element such that \( S\vec{v} = 0 \) is \( \vec{v} = 0 \); hence this implies that

\[
(T - T')\vec{u} = 0
\]

for all \( \vec{u} \in U \), i.e. \( T = T' \), as desired.

Conversely, suppose that \( S \) is not injective, so that \( \ker(S) \neq \{0\} \subset V \). This implies that there exists a nonzero linear map \( A : U \to V \) such that \( \{0\} \neq \text{Im}(A) \subset \ker(S) \). For instance, pick a basis \( \{\vec{u}_1, \ldots, \vec{u}_n\} \) for \( U \), let \( \vec{v} \) be an arbitrary nonzero vector in \( \ker(S) \), and define

\[
A\vec{u}_1 = \cdots = A\vec{u}_n = \vec{v}.
\]

Then, given any \( T \) such that \( R = ST \), we can construct another such map by defining \( T' := T + A \). Since \( \text{Im}(A) \subset \ker(S) \) by construction, we have \( SA = 0 \), so that

\[
ST' = S(T + A) = ST + 0 = R,
\]

as desired. And since \( A \) was not the zero map, we know that \( T' \neq T \), which finishes the proof.

\[ \square \]

Question 4. Let \( n \) be a positive integer and let \( \gamma = a + bi \) be a complex number with \( b \neq 0 \). Let \( V = \{p \in P(\mathbb{R}) : p(\gamma) = 0, \deg(p) \leq n\} \) be a subset of \( P(\mathbb{R}) \).

(a) Show that \( V \) is a subspace of \( P(\mathbb{R}) \).

Proof. We need to show the 3 properties of a subspace.

Additive identity: the zero polynomial, \( \vec{0}(x) = 0 + 0x + \cdots \) certainly has degree \( \leq n \) by definition, and also satisfies \( \vec{0}(\gamma) = 0 \); hence it is in \( V \).

Closed under addition: suppose \( p(x), q(x) \in V \). Then

\[
(p + q)(\gamma) = p(\gamma) + q(\gamma) = 0.
\]

Since \( \deg(p), \deg(q) \leq n \), they can be written as

\[
p(x) = a_0 + a_1x + \cdots + a_nx^n, \quad q(x) = b_0 + b_1(x) + \cdots + b_nx^n;
\]

Then we get that

\[
(p + q)(x) = (a_0 + b_0) + \cdots + (a_n + b_n)x^n,
\]

so that \( \deg(p + q) \leq n \) as well. This shows that \( p + q \subset V \), as desired.

Closed under multiplication: If \( p(x) \in V \) and \( c \) is a constant, then

\[
(cp)(\gamma) = c \cdot p(\gamma) = 0.
\]

Furthermore, if \( p(x) = a_0 + \cdots + a_nx^n \), then

\[
(cp)(x) = ca_0 + \cdots + ca_nx^n,
\]

so its degree is \( \leq n \) and hence \( cp \in V \) as well.

\[ \square \]
(b) Give a basis of $V$. What is its dimension?

**Proof.** Note that $V$ consists of polynomials with real coefficients. Therefore, by Theorem 4.10, $\bar{\gamma}$ is also a root of any $p(x) \in V$, and $\bar{\gamma} \neq \gamma$ since $b \neq 0$.

Define the real constants

$$\alpha := -(\gamma + \bar{\gamma}) = -2a, \beta := \gamma \bar{\gamma} = a^2 + b^2,$$

so that

$$(x - \gamma)(x - \bar{\gamma}) = x^2 + (-\gamma - \bar{\gamma})x + \gamma \bar{\gamma} = x^2 + \alpha x + \beta.$$

Now, by Prop. 4.1, we know that every $p(x) \in V$ can be expressed in the form

$$p(x) = (x^2 + \alpha x + \beta)p^*(x), \quad (1)$$

for some real polynomial $p^*(x)$ of degree $\deg(p) - 2$.

This implies that if $n \leq 1$, then $V$ consists of just the zero vector (the only polynomial with negative degree). Hence, if $n \leq 1$, the empty set is a basis for $V$ which has dimension 0.

If $n \geq 2$, then we claim $\dim(V) = n - 1$, and a basis is given by

$$\{(x - \gamma)(x - \bar{\gamma}), (x - \gamma)(x - \bar{\gamma})x, \ldots, (x - \gamma)(x - \bar{\gamma})x^{n-2}\}.$$

Indeed, by writing $p^*(x) = a_0 + a_1 x + \cdots + a_{n-2} x^{n-2}$ and plugging this into (1), we get that any $p(x)$ can be written as a linear combination

$$p(x) = a_0(x^2 + \alpha x + \beta) + a_1(x^2 + \alpha x + \beta)x + \cdots + a_{n-2}(x^2 + \alpha x + \beta)x^{n-2},$$

so that the vectors span $V$.

To see that the polynomials are independent, suppose that we have an arbitrary linear combination which is equal to 0:

$$a_0(x^2 + \alpha x + \beta) + \cdots + a_{n-2}(x^2 + \alpha x + \beta)x^{n-2} = (x^2 + \alpha x + \beta)(a_0 + \cdots + a_{n-2}x^{n-2}) = 0.$$

Then this implies that $a_0 + \cdots + a_{n-2}x^{n-2}$ is the zero polynomial, which only happens when $a_0 = \cdots = a_{n-2} = 0$, by Theorem 4.4.

This shows that the set of vectors above is linearly independent and spans $V$, hence is a basis for $V$ and gives the correct dimension, $n - 1$. 

$\square$