Math 113 Homework 5

Fall 2013

This homework is due Thursday November 7th at the start of class. Remember to write clearly, and justify your solutions. Please make sure to put your name on the first page and to number and staple your pages.

Problems From The Book: Axler Chapter 5 problems 1, 8, and 15 (Hint: see how the book proves the existence of eigenvectors), Chapter 8 problems 5 and 10. In addition do the following problems:

Question 1. Let $T_1, T_2 \in \mathcal{L}(\mathbb{C}^2)$ be given by the matrices

$$T_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, T_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$ 

What are the invariant subspaces of $T_1$ and $T_2$?

Question 2. Let $T \in \mathcal{L}(P_n(\mathbb{C}))$ be given by

$$T(p) = p + p' + p''.$$ 

What is the Jordan form of $T$? (Hint: show that $(T-I)^{n+1} = 0$ and that $(T-I)^n$ is not)

Question 3. Let $S, T \in \mathcal{L}(V)$ be operators on a finite dimensional, non-zero vector space $V$ over $\mathbb{C}$. Suppose that $S$ and $T$ commute, namely that $ST = TS$. Show that there is a non-zero vector $\vec{v} \in V$ so that $\vec{v}$ is an eigenvector of both $S$ and $T$. (Hint: Find a non-zero eigenvector of $T$, show that the eigenspace is $S$-invariant, and find an eigenvector of $S$ within that eigenspace).

Question 4. Let $T \in \mathcal{L}(\mathbb{C}^n)$ be an operator on $\mathbb{C}^n$. Suppose that $\mathbb{C}^n$ has a basis of eigenvectors of $T$, all of whose eigenvalues have absolute value strictly less than 1. Show that

$$\lim_{n \to \infty} T^n(\vec{v}) = \vec{0}$$

for all $\vec{v} \in \mathbb{C}^n$. You may use the fact that $\lim_{n \to \infty} \lambda^n = 0$ if $|\lambda| < 1$.

Question 5. Let $N \in \mathcal{L}(V)$ be a nilpotent operator. Let $p(x)$ be a polynomial. Show that $p(N)$ is nilpotent if and only if $p(0) = 0$.

Question 6. Approximately how much time did you spend working on this homework?