This homework is due Thursday October 24th at the start of class. Remember to write clearly, and justify your solutions. Please make sure to put your name on the first page and to number and staple your pages.

**Problems From The Book:** Axler Chapter 3 problems 3, 8, and 22, Chapter 4 problems 2 and 5. In addition do the following problems:

**Question 1.** (worth 50 points) Given a linear transformation $T \in \mathcal{L}(V, W)$ we know that the dimension of the image of $T$ is determined by the dimensions $V$ and of $\ker(T)$. Therefore, up to isomorphism $\operatorname{Im}(T)$ is determined by $V$ and $\ker(T)$. This makes sense because $T$ maps each translate of $\ker(T)$ to a point of $W$.

In the following extended question we will develop a rigorous theory expanding upon this idea.

Let $U$ be a subspace of $V$ (which we will think of as $\ker(T)$ above). Define a coset of $U$ to be a subset of $V$ of the form:

$$U + \vec{v} = \{\vec{u} + \vec{v} : \vec{u} \in U\}$$

for some $\vec{v} \in V$. In other words a coset is a translate of the subspace $U$.

(a) Show that $U + \vec{v} = U + \vec{v}'$ if and only if $\vec{v} - \vec{v}' \in U$.

We can define addition and scaling operations on cosets as follows:

$$(U + \vec{v}) + (U + \vec{w}) = (U + \vec{v} + \vec{w})$$

and

$$a(U + \vec{v}) = U + (a\vec{v})$$

(b) Show that the above operations are well defined. In particular show that if $\vec{v}$ or $\vec{w}$ is replaced by a different vector that gives the same coset ($(U + \vec{v}) = (U + \vec{v}')$), then the resulting sum or scaling yields the same coset.

Finally, we can use this operations to define the quotient space $V/U = \{\text{cosets of } U\}$ with addition and scaling operations as above.

(c) Show that $V/U$ with the above operations is a vector space.

There is a particularly nice function $\pi : V \to V/U$ given by $\pi(\vec{v}) = U + \vec{v}$.

(d) Show that $\pi$ is a linear transformation.

(e) Show that $\ker(\pi) = U$.

(f) Show that $\operatorname{Im}(\pi) = V/U$.

(g) Show that $\dim(V/U) = \dim(V) - \dim(U)$.

This $V/U$ turns out to be naturally isomorphic to the image of $T$. In particular, let $T \in \mathcal{L}(V, W)$ be a linear transformation with $U = \ker(T)$.

(h) Show that there is a unique $S \in \mathcal{L}(V/U, W)$ so that $T = S \circ \pi$.

(i) Show that $S$ is an isomorphism between $V/U$ and $\operatorname{Im}(T)$.

**Question 2.** Approximately how much time did you spend working on this homework?