1. Read §1–§6 in the handout on absolute values.

2. Compute the initial terms of the $p$-adic expansion (up to $aq_p^4$) for $1/7$ and $-3$ in $\mathbb{Q}_p$ for $p = 2, 3, 5$, where the expansions are taken to be in the form $\sum a_n p^n$ with $a_n \in \mathbb{Z}, 0 \leq a_n < p$. Also compute a square root of $2$ in $\mathbb{Q}_p$ for $p \in \{5, 7, 17\}$ up to the term $aq_p^4$, or explain why none exists; be clear on your choice of square root.

3. Let $F$ be a field that is complete with respect to a non-trivial discretely-valued absolute value $| \cdot |$, and let $A$ be the associated valuation ring (so $A$ is a discrete valuation ring). Let $F'/F$ be a finite extension (possibly inseparable!), and let $A'$ be the valuation ring of $F'$ with respect to the unique absolute value $| \cdot |'$ extending $| \cdot |$. Let $k$ and $k'$ be the respective residue fields of $A$ and $A'$.

   (i) Prove that if $\{a_i'\}$ is a finite set of elements of $A'$ that have $k$-linearly independent images in $k'$, then the $a_i'$s are $F$-linearly independent in $F'$. Conclude that $[k' : k] \leq [F' : F]$, so $f = [k' : k]$ is finite. This is called the residue field degree attached to $F'/F$.

   (ii) Using the norm-formula for $| \cdot |'$ in terms of $| \cdot |$, deduce that $| \cdot |'$ is discretely-valued, with $e \overset{\text{def}}{=} |[F'/\mathbb{Q}] : [F' : F]| \leq [F' : F]$, so $A'$ is a discrete valuation ring and $e$ is finite; this is called the ramification degree attached to $F'/F$. To justify the terminology, show that if $\pi \in A$ is a uniformizer and $\pi' \in A'$ is a uniformizer then $\pi = \pi'e^u$ with $u' \in A'^\times$.

   (iii) Let $\mathfrak{m}$ and $\mathfrak{m}'$ denote the maximal ideals of $A$ and $A'$. Choose $\{a_i'\}$ in $A'$ lifting a $k$-basis of $k'$. Prove that the $ef$ elements $a_i'\pi'^j$ for $1 \leq i \leq e$ and $0 \leq j \leq e - 1$ are $A$-linearly independent, and use $\mathfrak{m}'$-adic completeness of $A'$ and $\mathfrak{m}$-adic completeness of $A$ to prove that the $A$-linear inclusion

   $$\bigoplus_{1 \leq i \leq f, 0 \leq j \leq e - 1} Aa_i'\pi'^j \to A'$$

   is an isomorphism (hint: first study $A'/\mathfrak{m}^r$ for $1 \leq r \leq e$). Hence, $A'$ is a finite $A$-module (so $A'$ is the integral closure of $A$ in $F'$) and $[F' : F] = ef$.

4. Let $K$ be a field equipped with a non-trivial non-archimedean absolute value $| \cdot |$. Let $A$ be the valuation ring and $\mathfrak{m}$ its maximal ideal (so $\mathfrak{m} \neq 0$ since $| \cdot |$ is non-trivial).

   (i) Prove that $A$ is open in $K$ and $\mathfrak{m}$ is open in $A$. Also prove that for any $c \in K^\times$, $\{x \in K \mid |x| \leq |c|\}$ is open and closed, and is homeomorphic to $A$ via $x \mapsto x/c$. Conclude that $K$ is locally compact if and only if $A$ is compact.

   (ii) Assuming that $A$ is compact, deduce that $\mathfrak{m}$ has finite index in $A$, and hence that the residue field $k = A/\mathfrak{m}$ is finite. Also prove that in such cases $| \cdot |$ must be discretely-valued (so $A$ is a discrete valuation ring) and complete. Finally, conclude in general that $K$ is locally compact if and only if $K$ is complete and $k$ is finite.

   (iii) Now assume $K$ is locally compact and let $g = \#k$. Let the normalized absolute value $| \cdot |_K$ be the unique power of $| \cdot |$ with value group $\mathbb{Q}^\times$; that is, $| \cdot |_K = q^{-\text{ord}_K}$ where $\text{ord}_K : K^\times \to \mathbb{Z}$ is the normalized order function (sending uniformizers to 1). Prove that if $\mu$ is a Haar measure on $K$ (which makes sense since $K$ is locally compact) and $a \in K^\times$, then $\mu(aS) = \mu(S)$ (for Borel sets $S$) is a Haar measure on $K$ and $\mu_a = |a|_K \cdot \mu$. In other words, the normalized absolute value computes the scaling effect by $K^\times$ on Haar measures of $K$.

5. Continuing with the preceding exercise, let $K$ be a field that is locally compact with respect to a non-trivial non-archimedean absolute value. Our aim is to classify all such $K$. (The archimedean case was handled in the handout on absolute values.) By Exercise 4, $K$ is complete and its valuation ring $(A, \mathfrak{m})$ is a discrete valuation ring with residue field $k$ that is finite, say with size $q$. Let $p$ denote the characteristic of $k$.

   (i) Assume $\text{char}(K) > 0$. Prove that $K$ must have characteristic $p$, and prove that the algebraic closure $k_0$ of $\mathbb{F}_p$ in $K$ injects into $K$. Use Hensel’s Lemma to prove that in fact $k_0 \to k$ is an isomorphism, so via the inverse (Teichmüller lifting) we may canonically view $k$ as a subfield of $K$ (and hence $A$ also has a canonical structure of $k$-algebra).
(ii) Continuing with (i), upon choosing a uniformizer $\pi$ of $A$ prove that there exists a unique continuous map of $k$-algebras $\phi_\pi : k[[T]] \to A$ sending $T$ to $\pi$ (where $k[[T]]$ is given its $T$-adic topology), and that this map is an isomorphism (Hint: use completeness for $K$ to prove surjectivity). Conclude that there is an isometry $K \simeq k((T))$ over $k$ when we use the normalized absolute values on both sides. Explain conversely why $k((T))$ with its $T$-adic topology is locally compact for any finite field, so this gives a classification (up to isomorphism) of local fields with positive characteristic. (It is a general theorem in commutative algebra that if $(A, m)$ is any discrete valuation ring whose fraction field $F$ is $m$-adically complete with positive characteristic then there exists an abstract isomorphism of rings $A \simeq k[[T]]$ with $k$ the residue field of $A$, but if $k$ is not algebraic over $F_p$ then this $k$-algebra structure on $A$ is not canonical.)

(iii) Now assume $K$ has characteristic 0, so $p \in m$ is nonzero. Let $e = \text{ord}_K(p) > 0$ and let $f = [k : F_p]$. Let $\pi$ be a uniformizer of $A$. Explain why $K$ is canonically an extension of $\mathbb{Q}_p$, with $| \cdot |_K$ restricting to $| \cdot |_{F_p}$, and prove that if $\{a_i\}$ in $A$ is a set with $F_p$-linearly independent image in $k$ then the $a_i$’s are $\mathbb{Q}_p$-linearly independent in $K$.

(iv) Continuing with (iii), let $\{a_i\}$ in $A$ lift an $F_p$-basis of $k$. Prove that $\pi^e = pu$ with $u \in A^\times$, and deduce (via $\pi$-adic completeness of $A$ and $p$-adic completeness of $\mathbb{Z}_p$) that the natural $\mathbb{Z}_p$-linear map

$$\bigoplus_{1 \leq i \leq f, 0 \leq j \leq e-1} a_i \pi^j \mathbb{Z}_p \to A$$

is an isomorphism. Conclude that $[K : \mathbb{Q}_p] = ef$ is finite. Conversely, prove that any finite extension of $\mathbb{Q}_p$ (endowed with its canonical topology using the absolute value extending that on $\mathbb{Q}_p$) is a locally compact field. Hence, the non-archimedean local fields with characteristic 0 are precisely the finite extensions of the $p$-adic fields $\mathbb{Q}_p$. 