1. Some classical motivation

Let $A$ be a commutative ring. We have defined the Zariski topology on the set $\text{Spec}(A)$ of primes ideals of $A$ by declaring the closed subsets to be those of the form

$$V(I) = \{ p \supseteq I \}.$$ 

This is reminiscent of the classical situation where we worked with the set

$$k^n = \text{MaxSpec}(k[t_1, \ldots, t_n])$$

for an algebraically closed field $k$ (the equality with MaxSpec being essentially the Nullstellensatz) and declared the closed sets to be exactly the “affine algebraic sets”

$$Z(I) = \{ c \in k^n | f(c) = 0 \text{ for all } f \in I \} = \text{MaxSpec}(k[t_1, \ldots, t_n]/I).$$

In this handout, we will explore some topological and functorial features of $\text{Spec}(A)$ in general that are analogues or more easily visualized features in the classical setting. This suggests that one can study problems with rather general commutative rings by using a “geometric intuition” acquired from visualizations modeled on the classical case. Such a viewpoint is incredibly powerful, proving geometric insight even into purely number-theoretic problems such as the study of Diophantine equations (including an understanding of local and global obstructions to solutions to such equations).

We first wish to understand the functoriality of $\text{Spec}$ in $A$. As a warm-up, we consider the classical case:

**Example 1.1.** Let $k$ be an algebraically closed field, and let $Z \subset k^n$ and $Z' \subset k^m$ be Zariski-closed subsets. Let $f: Z \to Z'$ be a polynomial map, and let $f^*: k[Z'] \to k[Z]$ be the associated $k$-algebra map (defined by $f^*(h) = h \circ f$ in terms of $k$-valued functions). Then under the identifications

$$Z = \text{MaxSpec}(k[Z]), \quad Z' = \text{MaxSpec}(k[Z'])$$

provided by the Nullstellensatz (denoted as $z \mapsto m_z$ and $z' \mapsto m_{z'}$), we claim that $f$ can be reconstructed from $f^*$ as follows:

$$f(m) = (f^*)^{-1}(m).$$

That is, $m_{f(z)} = (f^*)^{-1}(m_z)$ for all $z \in Z$.

To prove this, we simply compute. An element $h \in k[Z']$ lies in $m_{f(z)}$ if and only if $h(f(z)) = 0$. But $h(f(z)) = (h \circ f)(z) = (f^*(h))(z)$, so this vanishes if and only if $f^*(h) \in m_z$, which is to say $h \in (f^*)^{-1}(m_z)$.

Now we turn things around for $\text{Spec}$ by taking the recipe in the preceding example as the *definition* of functoriality for $\text{Spec}$. More specifically, for a general ring map $\varphi: A \to B$ and a prime ideal $p$ of $B$, the preimage $\varphi^{-1}(p)$ is a prime ideal of $A$ (since the quotient ring $A/\varphi^{-1}(p)$ is a subring of the domain $B/p$ and hence is a domain).

Beware that MaxSpec is *not* similarly behaved for general rings, in contrast with finitely generated algebras over a field $k$ (and $k$-algebra maps!). For example, preimage under the injective map $k[T] \to k(T)$ carries the maximal ideal $(0)$ of $k(T)$ to the non-maximal prime ideal $(0)$ of $k[T]$. Note that in this example $k(T)$ is not finitely generated as a $k$-algebra. For an arbitrary field $k$, not necessarily algebraically closed, MaxSpec is well-behaved on finitely generated $k$-algebras precisely because of the Nullstellensatz. To be precise, consider a $k$-algebra map $\varphi: A \to B$ between such $k$-algebras, and $m$ a maximal ideal of $B$. The $k$-algebra domain $A/\varphi^{-1}(m)$ is a $k$-subalgebra of the field $B/m$ that is finite-dimensional over $k$ (Nullstellensatz!), forcing $A/\varphi^{-1}(m)$ to be finite-dimensional over $k$ and hence also a field. Thus, $\varphi^{-1}(m)$ is maximal as desired. We are now led to:

**Proposition 1.2.** For a ring map $\varphi: A \to B$, the induced map of sets $X(\varphi): \text{Spec} B \to \text{Spec} A$ defined by $p \mapsto \varphi^{-1}(p)$ is continuous, and $A \rightsquigarrow \text{Spec} A$ is a contravariant functor from the category of rings to the category of topological spaces.
Proof. The contravariant functoriality is straightforward: if \( \varphi \) is the identity then \( X(\varphi) \) is clearly the identity map, and if \( \psi : B \to C \) is another ring map then \( X(\varphi) \circ X(\psi) \) carries a prime ideal \( q \) of \( C \) to \( \varphi^{-1}(\psi^{-1}(q)) = (\psi \circ \varphi)^{-1}(q) = X(\psi \circ \varphi)(q) \). That is, \( X(\varphi) \circ X(\psi) = X(\psi \circ \varphi) \).

For the continuity, it suffices to check that the preimage of a closed set is closed. This is a computation: \( \varphi^{-1}(V(I)) = V(\varphi(I)B) \) for any ideal \( I \) of \( B \). That is, if \( p \) is a prime ideal of \( A \) then we claim that \( X(\varphi)(p) \in V(I) \) if and only if \( p \) contains \( \varphi(I)B \), or equivalently if and only if \( \varphi(I) \subseteq p \). This latter containment says exactly that \( I \subseteq \varphi^{-1}(p) \), and that in turn is precisely the statement that \( X(\varphi)(p) \in V(I) \).

2. Topological interpretation of algebra

As applications of the functoriality of \( \text{Spec} \), we can identify certain closed or open subsets of \( \text{Spec} \ A \) with the spectrum of certain quotients or localizations of \( A \):

**Proposition 2.1.** Let \( I \subseteq A \) be an ideal, and \( a \in A \) be an element. The continuous map \( \text{Spec}(A/I) \to \text{Spec}(A) \) induced by the natural map \( A \to A/I \) is a homeomorphism onto \( V(I) \), and the continuous map \( \text{Spec}(A_a) \to \text{Spec}(A) = X \) induced by the natural map \( A \to A_a \) is a homeomorphism onto the open set \( X_a = X - V((a)) \).

Proof. The prime ideals of \( A/I \) are precisely \( p/I \) for primes \( p \in V(I) \), and the preimage of \( p/I \) under \( A \to A/I \) is exactly \( p \), so \( \text{Spec}(A/I) \) is certainly carried bijectively onto \( V(I) \). To see that this continuous bijection is a homeomorphism, we compute with closed sets. A closed set in \( \text{Spec}(A/I) \) is the set of primes containing an ideal \( J \), and \( J = J/I \) for an ideal \( J \) of \( A \) containing \( I \). Thus, \( V(J) \) makes sense as a closed subset of \( V(I) \), and it is easy to see (check!) that the bijection \( \text{Spec}(A/I) \to V(I) \) identifies \( V(J) \) with \( V(J) \).

The prime ideals of \( A_a \) are precisely \( pA_a \) for primes \( p \) of \( A \) that do not contain \( a \), and under the map \( A \to A_a \) the preimage of \( pA_a \) is exactly \( p \) (why?), so \( \text{Spec}(A_a) \) is carried bijectively onto \( X_a \). To see that this continuous bijection is a homeomorphism, we again compute with closed sets. A closed set in \( \text{Spec}(A_a) \) is the set \( V_{A_a}(I) \) of primes containing an ideal \( I \) of \( A_a \), and \( J = I \cdot A_a \) where \( I \subseteq A \) is the “ideal of numerators” of \( J \) (i.e., \( I \) is the set of \( a' \in A \) such that \( a'/a^n \in J \) for some \( n \geq 0 \)). We claim that the closed set \( X_a \cap V(I) \) in \( X_a \) is identified with \( V_{A_a}(I \cdot A_a) \) under the continuous bijection \( \text{Spec}(A_a) \to X_a \). This says exactly that for a prime \( p \) of \( A \) not containing \( a \), \( p \) contains \( I \) if and only if the prime ideal \( p \cdot A_a \) of \( A_a \) contains \( I \cdot A_a \). The implication “\( \Rightarrow \)” is obvious, and for the converse we note that if \( I \cdot A_a \subseteq p \cdot A_a \) then for every \( a' \in I \) necessarily \( a' = x/a^n \) in \( A_a \) with some \( x \in p \) and some \( n \geq 0 \). But then \( a^n a' \) and \( x \) coincide in \( A_a \), so for some \( m \geq 0 \) the equality \( a^{n+m} a' = a^n x \) holds in \( A \), so \( a^{n+m} a' \in p \). Since \( a \not\in p \) and \( p \) is prime, it follows that \( a' \in p \) as desired.

We can also interpret the irreducible closed sets in terms of prime ideals, analogous to the dictionary between irreducible affine algebraic sets and prime ideals in the classical setting:

**Proposition 2.2.** The irreducible closed sets in \( X = \text{Spec} A \) are exactly \( V(p) \) for prime ideals \( p \) of \( A \), and each irreducible closed set \( Z = V(p) \) contains \( p \) as its unique dense point.

We sometimes call the dense point \( \{p\} \) the generic point of \( V(p) \). Loosely speaking, we can visualize \( \text{Spec}(k[t_1, \ldots, t_n]) \) for \( k = \overline{k} \) as being obtained from \( \text{MaxSpec}(k[t_1, \ldots, t_n]) = k^n \) by adding in a new point \( \{p\} \) for every classical irreducible closed set \( Z(p) \), with this new point having closure whose closed points are precisely the points of the classical irreducible closed set of interest. (For example, in \( k^3 \) an irreducible surface acquires its own new dense point as well as a new generic point on every irreducible curve in the surface.)

Proof. Every irreducible closed set has the form \( V(I) \) for a unique radical ideal \( I \) of \( A \), and prime ideals are certainly radical, so the first assertion is that for radical \( I \) the closed set \( V(I) \) is irreducible if and only if \( I \) is prime. Since \( V(I) \) is homeomorphic to \( \text{Spec}(A/I) \), we can express everything in terms of the quotient ring \( A' = A/I \): if \( A' \) is a reduced ring (i.e., no nonzero nilpotents) then \( \text{Spec} A' \) is irreducible if and only if \( A' \) is a
domain. To prove this equivalence, we will imitate some of the calculations used in the proof of its classical counterpart (that $\mathbb{Z}(J)$ is irreducible if and only if $J$ is prime, where $J$ is a radical ideal of $k[t_1, \ldots, t_n]$).

Suppose first that $\text{Spec}(A')$ is irreducible (so it is non-empty, and hence $A' \neq 0$). To show that $A'$ is a domain, we consider $a, b \in A'$ such that $ab = 0$ and we want to show that $a = 0$ or $b = 0$ in $A'$. Every prime contains $0 = ab$ and hence contains either $a$ or $b$. That is, $\text{Spec}(A') = V((a)) \cup V((b))$. This expresses the irreducible $\text{Spec}(A')$ as a union of two closed subsets, so one of these subsets must be the entire space. That is, either $V((a))$ or $V((b))$ coincides with $\text{Spec}(A')$, which is to say that either $a$ lies in every prime or $b$ does. However, the intersection of all prime ideals is the nilradical (set of nilpotent elements), which in $A'$ is $(0)$ since $A'$ is assumed to be reduced. Hence, $(a) = (0)$ or $(b) = (0)$, so we get the vanishing of $a$ or $b$. Conversely, if $A'$ is a domain (hence nonzero) then $\text{Spec}(A')$ is certainly non-empty and so to show it is irreducible we just need to show that if a pair of closed sets $Z, Z'$ cover the entire space then one of them is the entire space. In fact, in such cases there is the point $\eta = \{(0)\}$ (since $A'$ is a domain!) that I claim is dense. Granting this, the dense point must lie in one of the closed sets $Z$ or $Z'$ that are assumed to cover the entire space, but then by density of this point we see that whichever of the closed sets $Z$ or $Z'$ contain this point must in fact be the entire space. It remains (for the characterization of irreducible closed sets in terms of prime ideals) to show that $\{(0)\}$ is dense in $\text{Spec}(A)$ when $A$ is a domain. In other words, we claim that the only closed set $Z$ which contains $\{(0)\}$ is the entire space. Indeed, if $Z = V(I)$ contains $\{(0)\}$ then by definition $I \subseteq (0)$, so $I = (0)$ and hence $Z = V((0))$ is the whole space.

Now we turn to the other assertion: $\{p\}$ is the unique dense point in $V(p)$. Since $V(p)$ is closed in the entire space, this assertion is intrinsic to the topological space $V(p)$. Under the identification of $V(p)$ with $\text{Spec}(A)$ as topological spaces, we can rename $A/p$ as $A$ and replace $p$ with $(0)$ to reduce to the following claim: if $A$ is a domain then $\{(0)\}$ is the unique dense point of $\text{Spec}(A)$. We saw above that this point is dense, so it remains to show that it is the only dense point. That is, if $p$ is a nonzero prime of the domain $A$ then we claim that the closure of $\{p\}$ is not the entire space. But we have seen that the closure is exactly $V(p)$, so we just have to check that if $p \neq (0)$ then $V(p) \neq \text{Spec}(A)$. But this is clear, as the point $\{(0)\}$ is certainly not in $V(p)$ when $p \neq (0)$ (why?).