Math 210A. Nakayama’s Lemma

Let $A$ be a local ring with unique maximal ideal $\mathfrak{m}$, and let $M$ be a finitely generated $A$-module. The aim of this handout is to prove the extremely useful Nakayama’s Lemma: a subset of $M$ is a spanning set over $A$ if and only if its image in the $A/\mathfrak{m}$-vector space $M/\mathfrak{m}M$ is a spanning set over $A/\mathfrak{m}$. (Taking the potential spanning set $\{0\}$, it follows as a special case that $M = 0$ if and only if $M/\mathfrak{m}M = 0$. We will actually deduce the general case from this special case.)

Beware that the finite generation hypothesis is crucial. For a counterexample otherwise, consider the local ring $A = \mathbb{Z}(p)$ and the quotient $A$-module $M = \mathbb{Q}/\mathbb{Z}(p)$. In this case $\mathfrak{m} = pA$, so $M/\mathfrak{m}M = M/pM = 0$ (since every element of $\mathbb{Q}$ has the form $px$ for some $x \in \mathbb{Q}$). However, obviously $M \neq 0$ (and also $M$ is not finitely generated as an $A$-module in this case).

To prove Nakayama’s Lemma, let $N \subseteq M$ be the submodule spanning by the given subset, so by hypothesis $N$ maps onto $M/\mathfrak{m}M$. We wish to conclude that $N = M$ (i.e., $M/\mathfrak{N} = 0$). By passing to the (finitely generated!) quotient module $M/N$ and renaming it as $M$, we reduce to the following special case: if $M/\mathfrak{m}M = 0$ then $M = 0$. That is, if $M$ is finitely generated over the local ring $A$ and $M = \mathfrak{m}M$ then we wish to conclude that $M = 0$. To prove this, we pick a finite spanning set $\{x_1, \ldots, x_n\}$ of $M$ (as we may, by the finite generation hypothesis) and we shall proceed by induction on $n$. If $n = 1$ then $M = Ax_1$, so $\mathfrak{m}M = \mathfrak{m}x_1$. Hence, all elements of $M$ have the form $ax_1$ for some $a \in \mathfrak{m}$. In particular, $x_1 = ax_1$ for some such $a$. It follows that $(1 - a)x_1 = 0$ in $M$. But $1 - a$ does not lie in the unique maximal ideal of $A$ (since $a \in \mathfrak{m}$), so it must be a unit! Hence, multiplying by $(1 - a)^{-1}$ makes sense, so $x_1 = 0$ in $M$. But $x_1$ spans $M$, so $M = 0$ as desired. This settles the case $n = 1$.

Now assume $n > 1$ and that the result is proved whenever there are $n - 1$ generators. Since $M = \mathfrak{m}M$, every element of $M$ has the form $\sum a_ix_i$ with $a_i \in \mathfrak{m}$. Applying this to $x_n$, we get $x_n = \sum a_ix_i$ for some $a_i \in \mathfrak{m}$, so $(1 - a_n)x_n = \sum_{i<n} a_ix_i$. But $1 - a_n \notin \mathfrak{m}$ since $a_n \in \mathfrak{m}$, so by locality $1 - a_n \in A^\times$. Hence, we can scale by $(1 - a_n)^{-1}$ to deduce that $x_n$ is in the $A$-span of $x_1, \ldots, x_{n-1}$. It follows that $M$ is spanned over $A$ by $x_1, \ldots, x_{n-1}$, so by induction on $n$ we get $M = 0$ as desired.