Math 210A. Euler characteristic

Let $M^\bullet$ be a finite complex of finite-dimensional vector spaces over a field $k$. In particular, not only are all $M^i$'s of finite dimension, but so are the homologies $H^i(M^\bullet)$ for all $i$. A basic observation of Euler is the identity

$$
\sum (-1)^i \dim M^i = \sum (-1)^i \dim H^i(M^\bullet).
$$

In this handout we prove Euler’s identity.

The key point is to “chop up” the given complex into many short exact sequences, and repeatedly apply the basic formula

$$
\dim V = \dim V' + \dim V''
$$

for any short exact sequence $0 \to V' \to V \to V'' \to 0$ of finite-dimensional $k$-vector spaces. More specifically, using the short sequences

$$
0 \to \ker d^i \to M^i \to \im(d^i) \to 0, \quad 0 \to \im(d^{i-1}) \to \ker d^i \to H^i(M^\bullet) \to 0
$$

we obtain

$$
\dim M^i = \dim \ker(d^i) + \dim \im(d^i), \quad \dim H^i(M^\bullet) = \dim \ker(d^i) - \dim \im(d^{i-1}),
$$

so

$$
\dim M^i - \dim H^i(M^\bullet) = \dim \im(d^i) + \dim \im(d^{i-1}).
$$

Multiplying through by $(-1)^i$ and summing over all $i$, the right side sums to

$$
\sum (-1)^i \dim \im(d^i) - (-1)^{i-1} \dim \im(d^{i-1}) = 0.
$$

This yields the desired identity.