Math 239 – Homework 1 – Due Monday, April 17 in class

Questions 1 through 4 involve coding. If possible, try to avoid loops because they slow down MATLAB. Instead, use MATLAB matrix functions. You may want to look at GBM.m that simulates the values of \( p \) Geometric Brownian motion paths evaluated at \( N \) time steps without a loop. Compare with GBM_loop.m that does the same thing.

**Question 1. Delta hedging simulation.**

The purpose of this exercise is to use simulation to measure the hedging error resulting from discrete rebalancing of a hedge. You sell a 3-month European call option at the Black-Scholes price and try to hedge it by holding “delta” shares of the stock. You can borrow money from the bank at constant interest rate. Any money left earns the same rate of interest.

At initiation of the contract, you get the premium from the client and buy “delta” shares of the stock. You may need to borrow extra money.

At each time step, the stock has evolved from the previous step and the “delta” must be adjusted. Depending on how it has changed, you need either to buy or sell shares. You also pay or earn interest on any money borrowed or deposited over the previous period.

At maturity, you close your position. This means: selling all shares you own, reimbursing the bank for the money you owe or get what is left in your account, and paying your client the amount \( \max(0, S_T - K) \). How much cash is left after that is your profit or loss.

Assume the underlying asset \( S \) is modeled by geometric Brownian motion and use parameters in Table 1. Black-Scholes theory says that, as the number of times you rebalance goes to infinity (with \( T \) fixed), your hedging error (profit or loss) goes to zero on each path. Since, in practice, continuous trading is impossible, the hedge is imperfect and we want to study this imperfection.

| initial price | \( S_0 = 50 \) | rate of return | \( \mu = 10\% \) |
| volatility    | \( \sigma = 30\% \) | interest rate | \( r = 5\% \) |
| strike        | \( K = 50 \) | expiration    | \( T = 0.25 \) |

Table 1: Parameters
(a) Find the mean and standard deviation of the hedging error with daily and
weekly rebalancing. Make a histogram of the distribution of hedging errors for
both cases. You may use the function `hist`.

(b) Take different values for $\mu$. How does that change the results? Is this surprising?

(c) Let $\Delta t$ denote the rebalancing interval. We know that as $\Delta t \to 0$ the hedging
error goes to 0; what does your simulation suggest about the rate of conver-
genence? Does the hedging error appear to be $O((\Delta t)^a)$ for some $a$, and if so,
what $a$? (You may want to draw a log-log plot.)

Question 2. Stop-loss start-gain strategy.

The purpose of this exercise is to test another hedging scheme for a call option. This
strategy consists first, in charging $(S_0 - Ke^{-rT})^+$ for the price of the option, and
then, in hedging it by holding one share when $S_t > K e^{-r(T-t)}$ and no share at all
when $S_t \leq K e^{-r(T-t)}$. It is based on the simple observation that the seller will need
one share at expiry if $S_T > K$ and none if $S_T \leq K$.

Modify your code in Question 1 and find the mean and standard deviation of the
hedging error with daily and weekly rebalancing. Make a histogram of the distribution
of hedging errors for both cases. Compare with the delta hedging strategy and explain
the difference.

Question 3. Transaction costs.

The purpose of the exercise is to see what happens if there are transaction costs.
The strategy here is exactly the same as in Question 1 except that you must also
pay your broker a fee per transaction that is a fraction $k$ of the dollar amount of the
transaction. For simplicity, this rule does not apply either at initiation or at expiry
but at each rehedging times. You may take $k = 2\%$.

(a) Repeat Question 1.(a). How do the results change in the presence of transaction
costs?

(b) H. Leland (in [1]) proposed the following strategy in presence of transaction
costs. He says that to make up for the additional costs, the call option must be
sold and hedged as in Black-Scholes, but with a higher volatility parameter \( \hat{\sigma} \):

\[
\hat{\sigma} = \sigma \sqrt{1 + \frac{8}{\pi} \frac{k}{\sigma \sqrt{\Delta t}}}.
\]

The stock keeps its volatility \( \sigma \), the modified volatility only plays a role in computing prices and hedges. Check that the Black-Scholes price computed with \( \hat{\sigma} \) is greater. Repeat Question 1.(a) with this strategy. Is it a good strategy?

**Question 4. Non constant volatility.**

The purpose of this exercise is to see what happens to the Black-Scholes strategy if the volatility is not constant. Let us suppose that the stock is a CEV (Constant Elasticity of Variance) process:

\[
dS_t = S_t \left[ \mu dt + \alpha S_t^\beta dW_t \right].
\]

\( \beta = 0 \) corresponds to geometric Brownian motion. Take \( \beta = -0.8 \) and \( \alpha \) such that \( \alpha S_0^\beta = 0.3 \) (that is, the volatility at initiation of the contract is also 30%).

To simulate \( N \) values of this process at times \( T/N, \cdots, T \), use the discretization scheme discussed in class:

\[
S_{(i+1)T/N} = S_{iT/N} \left[ 1 + \frac{\mu T}{N} + \alpha (S_{iT/N})^\beta Z_{i+1} \right] \quad i = 0, \cdots, N - 1.
\]

where the \( Z_i \)'s are i.i.d. Gaussian random variables with mean 0 and variance \( T/N \).

Modify your code in Question 1 but keep hedging your option with Black-Scholes delta at constant volatility 30%. Repeat Question 1.(c) and compare results with the constant volatility case.

**Question 5. Gaussian processes and Brownian bridge.**

For a given square integrable process \( X \), define the mean and covariance functions:

\[
m(t) = \mathbb{E} \{ X_t \} \quad \text{and} \quad K(s,t) = \mathbb{E} \{ (X_t - m(t))(X_s - m(s)) \}.
\]

Recall that a Gaussian process is a process \( X \) such that for every \( n \geq 1 \), for all positive reals \( t_1, \cdots, t_n \), and all reals \( a_1, \cdots, a_n \), \( \sum_{i=1}^n a_i X_{t_i} \) has the Gaussian distribution. Important fact: the law of a Gaussian process is completely characterized by its mean and covariance functions.
(a) Show that Brownian motion is a Gaussian process and show that its mean is 0 and its covariance function is \( \inf(s, t) \).

(b) Let \( W \) be Brownian motion and define \( X_t = W_t - tW_1 \) for \( 0 \leq t \leq 1 \). Show that \( X \) is a Gaussian process and compute its mean and covariance functions. Such a process is called a Brownian bridge (because \( X_0 = X_1 = 0 \)).

(c) Show that \((1-t)W_t/(1-t)\) defined for \( 0 \leq t < 1 \) is also a Brownian bridge. [Hint: first show that it is a Gaussian process.]

(d) Let \( Y \) be the solution to the following stochastic differential equation for \( 0 \leq t < 1 \):

\[
dY_t = -\frac{Y_t}{1-t}dt + dW_t
\]

and \( Y_0 = 0 \). Using Itô’s formula, show that

\[
Y_t = (1-t)\int_0^t \frac{dW_s}{1-s}.
\]

Show that \( Y \) is also a Brownian bridge. [Hint: you may use the fact that if \( f \) is a square integrable deterministic function, then \( \{\int_0^t f(s)dW_s : t \geq 0\} \) is a Gaussian process with mean 0 and covariance \( K(s, t) = \int_0^{\inf(s, t)} f(u)^2 du \).]

References