Math 240 – Homework 2 – Due Monday, May 8

Question 1. Black’s formula

Using the change of numéraire technique compute the price of a European call option on stock $S$ with strike $K$ and maturity $T$ when interest rates are allowed to be stochastic. Assume that the volatility of the forward price for delivery at time $T$, $F_t(T) = S_t/B_t(T)$, is constant.

Justify the use of the risk-neutral valuation rule by computing the replicating portfolio.

Question 2. ATM implied cap volatilities with Vasicek

(a) Derive the formula for a European call option on a zero-coupon bond in Vasicek’s model. Call option has maturity $T$, strike $K$, and the zero-coupon bond has maturity $U > T$. Today is time 0.

\[ p = B_0(U)\Phi \left( \frac{r^* - \ell_1(T, U, r_0)}{\sqrt{\ell_2(T)}} \right) - KB_0(T)\Phi \left( \frac{r^* - \ell_1(T, T, r_0)}{\sqrt{\ell_2(T)}} \right) \]

where $r^* = -(A(T, U) + \ln K)/C(T, U)$

\[ A(t, T) = \left( b - \frac{\sigma^2}{2a^2} \right) \left( T - t - \frac{1 - e^{-a(T-t)}}{a} \right) + \frac{\sigma^2}{4a^3} \left( 1 - e^{-a(T-t)} \right)^2 \]

\[ C(t, T) = 1 - e^{-a(T-t)} \]

\[ \ell_1(T, U) = re^{-aT} + b \left( 1 - e^{-aT} \right) - \frac{\sigma^2}{2a^2} \left( 2 - 2e^{-aT} - e^{-a(U-T)} + e^{-a(T+U)} \right) \]

\[ \ell_2(T) = \frac{\sigma^2}{2a} \left( 1 - e^{-2aT} \right) \]

(b) Plot the yield curve $R_0(T)$.

(c) Compute and plot the ATM cap implied volatilities for maturities 1, 2, \ldots, 30 years. Assume $\delta = T_0 = T_i - T_{i-1} = 1/4$. 

1
Parameters for Vasicek: $a = 0.86$, $b = 9\%$, $r_0 = 8\%$, and $\sigma = 1.48\%$.

Matlab is recommended for the last two questions. Once you have computed cap prices, you can use the function `ivcap.m` which will compute the corresponding Black's implied volatilities. It takes as inputs: a cap price, a vector of dimension $n+1$ containing $B(T_0), \ldots, B(T_n)$, the parameters $n, \delta$, and strike $\kappa$, and returns the implied volatility.

**Question 3. An example of a local-martingale that is not a martingale**

Let $W$ and $\widetilde{W}$ be two independent Brownian motions. Recall that

$$P\left\{ \exists t > 0 \ W_t^2 + \widetilde{W}_t^2 = 0 \right\} = 0.$$  

(The planar Brownian motion never comes back to the origin.) We can therefore define the process $X_t = \ln \left( W_t^2 + \widetilde{W}_t^2 \right)$ for $t \geq 1$. Show that $X$ is a local martingale but not a martingale. [Hint: Compute $E \{ X_t \}$.]