Short rate models.

We consider an economy where the primary securities are bonds $B_t(T)$ for every $T \geq t$. If we assume that there are no arbitrage opportunities, we saw that this meant the existence of a risk premium vector $\lambda_t$ such that the return on $B_t(T)$ write:

$$\frac{dB_t(T)}{B_t(T)} = r_t dt + \Gamma_t(T).dW_t + \lambda_t dt$$

$W$ is $d$-dimensional, $\lambda$ does not depend on $T$ and $\Gamma_t(T)$ is the vector of volatilities for the $T$-bond. $\Gamma_t(T)$ can be random but will assume that it is bounded.

If $\lambda_t$ satisfy a Novikov-type condition, we introduce the risk-neutral probability $Q$ and

$$\frac{dB_t(T)}{B_t(T)} = r_t dt + \Gamma_t(T).dW_t$$

where $W$ is $Q$-Brownian motion.

$$\mathbb{E}_t^Q \left\{ e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right\}$$

is a martingale because $\Gamma$ is bounded, and therefore, since $B_t(T)=1$:

Given this equation, it is natural to look for models for the short rate $r$ under the risk-neutral measure $Q$.

The Vasicek model: $d=1$ and

$$\begin{cases}
    dr_t = (a - r_t) dt - \sigma dW_t \\
    r_0 = r
\end{cases}$$

$\Gamma$ is an Ornstein-Uhlenbeck process: a Gaussian process which is mean reverting around a mean level $b$ and reversion speed $a$.

Solution:

$$r_t = r_0 e^{-at} + b (1 - e^{-at}) - \sigma \int_0^t e^{-a(t-s)} dW_s$$

To compute bond prices, we will need the law of $\int_0^T r_s ds$ given $r_t$.

Since $(r_t)$ is a Gaussian process, it will have the Gaussian distribution.

If we denote $I_{t,T} = \int_t^T r_s ds$, then

$$aI_{t,T} = -(r_T - r_t) + ab(T-t) - \sigma \int_t^T dW_s$$

and therefore

$$I_{t,T} = b(T-t) + (b - r_t) \frac{1 - e^{-a(T-t)}}{a} - \frac{1}{a} \int_t^T \frac{1 - e^{-a(T-s)}}{a} dW_s$$
From which, we easily compute:

\[ E \left\{ \int_t^T r_s \, ds \mid r_t \right\} = b(T-t) + (b-r_t) \frac{1-e^{-a(T-t)}}{a} \]

\[ \text{Var} \left\{ \int_t^T r_s \, ds \mid r_t \right\} = \sigma^2 \int_t^T \left( \frac{1-e^{-a(T-t)}}{a} \right)^2 \, ds = -\frac{\sigma^2}{2a^3} \left( 1 - e^{-a(T-t)} \right)^2 + \frac{\sigma^2}{a^2} \left( T-t - \frac{1-e^{-a(T-t)}}{a} \right) \]

We can compute:

\[ B_t(T) = \mathbb{E} \left[ e^{-\int_t^T r_s \, ds} \mid \mathcal{F}_t \right] = \exp \left\{ -b(T-t) + (b-r_t) \frac{1-e^{-a(T-t)}}{a} - \frac{\sigma^2}{4a^3} \left( 1 - e^{-a(T-t)} \right)^2 \right. \\
\left. + \frac{\sigma^2}{2a^2} \left( T-t - \frac{1-e^{-a(T-t)}}{a} \right) \right\} \]

\[ B_t(T) = \exp \left\{ -\frac{1-e^{-a(T-t)}}{a} r_t - R_\infty \left( T-t - \frac{1-e^{-a(T-t)}}{a} \right) - \frac{\sigma^2}{4a^3} \left( 1 - e^{-a(T-t)} \right)^2 \right\} \]

where \( R_\infty = b - \frac{\sigma^2}{2a^2} \)

The volatility is \( \Gamma_t(T) = \sigma \frac{1-e^{-a(T-t)}}{a} \)

The yield curve:

\[ R_t(T) = \frac{1-e^{-a(T-t)}}{a(T-t)} r_t + R_\infty \left( 1 - \frac{1-e^{-a(T-t)}}{a(T-t)} \right) + \frac{\sigma^2}{4} \left( \frac{1-e^{-a(T-t)}}{a(T-t)} \right)^2 \]

\[ \lim_{T \to \infty} R_t(T) = R_\infty \]

Sketch of possible yield curves:
Cox-Ingersoll-Ross model (1985) \( dX_t = \alpha (\bar{X} - X_t) \, dt - \sigma \sqrt{X_t} \, dW_t \)

is a process often called a mean-reverting square root process.

The square root guarantees that it stays non-negative. Moreover if \( 2\alpha \beta > \sigma^2 \), then
the process never reaches 0. This intuitively means that the pull back face is
sufficiently strong.

Then there is no explicit solution unlike Vasicek, but we can still compute \( B_t(T) = E(e^{-\int_0^T \sigma dW_t}) \)
explicitly. Let's look for \( B_t(T) = f(t, \frac{r}{2}, T) \)

\[
\frac{df}{dt} + \frac{dr}{dt} a(b-r) + \frac{\sigma^2 r}{2} \frac{dr}{dt} \right) dt - \sigma \sqrt{r} \frac{dr}{dt} dW_t
\]

Since \( e^{B_t(T)} \) is a martingale and \( B_t(T), \frac{r}{2} \) we get the following PDE with terminal condition:

\[
\left\{
\begin{aligned}
\frac{df}{dt} + a(b-r) \frac{df}{dr} + \frac{\sigma^2 r}{2} \frac{df}{dr^2} &= r f \\
f(t, r, T) &= 1
\end{aligned}
\right.
\]

We look for a solution of the type \( f(t, r, T) = e^{-A(t,T) - C(t,T)r} \).

Plugging in \(-A + A') f = a(b-r) C_f + \frac{\sigma^2 r}{2} C_f^2 f - r f = 0\)

\( A(T,T) = C(T,T) = 0 \).

Which leads to 2 ODE's:

\[
\left\{
\begin{aligned}
A' + ab C &= 0 \\
-C' + ab C + \frac{\sigma^2}{2} C^2 &= 0
\end{aligned}
\right. \quad \text{Ricatti}
\]

Whose solutions are:

\[
A(t, T) = \frac{2(e^{Y(T-t)}-1)}{(Y+a)(e^{Y(T-t)}-1)+2Y} \quad \text{with} \quad Y = \sqrt{a^2 + \sigma^2 t}
\]

\[
C(t, T) = \frac{2(e^{Y(T-t)}-1)}{(Y+a)(e^{Y(T-t)}-1)+2Y} \quad \text{with} \quad Y = \sqrt{a^2 + \sigma^2 t}
\]

The yield curve is: \( r_t(T) = \frac{A(t,T)}{T-t} + \frac{C(t,T)}{T-t} \frac{r_t}{T-t} \)

and the zero coupon bond price volatility is \( \Gamma_t(T) = \sigma C(t,T) \sqrt{r_t} \) (stochastic)
Affine models.

The previous 2 models (Vanicek and CIR) have the nice property that bond prices have explicit solutions as

$$B_t(T) = \exp \left( -A(t,T) - C(t,T) r_t \right)$$

This is the case for a certain class of diffusion models for $r_t$ called affine models. Duffie and Kan (1993) showed that a sufficient condition for that is:

$$d r_t = (b_t + \beta_t r_t) dt + \sqrt{a_t + \kappa_t r_t} dW_t$$

with $b, \beta, a, \kappa$ being deterministic functions of time.

$$B_t(T) = f(t, r_t, T)$$

with $f$ satisfying:

$$\begin{cases} \frac{\partial f}{\partial t} + (b_t + \beta_t r_t) \frac{\partial f}{\partial r} + \frac{1}{2} (a_t + \kappa_t r_t) \frac{\partial^2 f}{\partial r^2} - rf = 0, \\ f(T, r, T) = 1. \end{cases}$$

Again looking for a solution of the form: $B_t(T) = \exp \left( -A(t,T) - C(t,T) r_t \right)$ leads to:

$$\begin{cases} - \left( A' + C' r \right) - \left( b_t + \beta_t r_t \right) C + \frac{1}{2} \left( a_t + \kappa_t r_t \right) C^2 - r = 0, \\ A(T, T) = C(T, T) = 0. \end{cases}$$

$$A'(t) = -b_t C(t, T) + \frac{1}{2} a_t C^2(t, T),$$

$$A(T, T) = 0,$$

$$C'(t, T) = -\beta_t C(t, T) + \frac{a_t}{2} C^2(t, T) = 1 \leftarrow \text{Riccati's equation.}$$

$$C(T, T) = 0$$

First solve for $C$ from the second equation and then solve the first.

In general there are no explicit solutions but solving ODE's has a well-known numerical solutions.

The zero coupon bond price volatility is

$$\Gamma_t(T) = C(t, T) \sqrt{a_t + \kappa_t r_t}$$

Examples:

- Ho-Lee model
  $$d r_t = b_t dt + \sigma dW_t$$

- Extended Vanicek
  $$d r_t = (b_t + \beta_t r_t) dt + \sigma dW_t$$

- Extended CIR
  $$d r_t = (b_t + \beta_t r_t) dt + \sigma \sqrt{r_t} dW_t.$$
Let us consider an example: bond option in Varieck.

Bond has maturity $U$ and the option has maturity $T$ and strike $K$. $t \leq T < U$.

$$P_t = E^Q \left\{ e^{-\int_t^T r_s \, ds} (B_T(U) - K)^+ \bigg| F_t \right\} \text{ in its price at time } t.$$ 

Since Varieck is an affine model, $B_T(U) = e^{-A(T, U) - C(T, U) r_t}$ when $A$ and $C$ are deterministic functions that we already computed.

The "exercise region" $\{ B_T(U) > K \} = \{ r_T \leq r^* \}$ because $C$ is positive.

with $r^* = \frac{-A(t, U) + \ln K}{C(t, U)}$

Then $$P_t = E^Q \left\{ e^{-\int_t^U r_s \, ds} B_T(U) 1\{r_T \leq r^*\} \bigg| F_t \right\} - K E^Q \left\{ e^{-\int_t^T r_s \, ds} 1\{r_T \leq r^*\} \bigg| F_t \right\}$$

$$= E^Q \left\{ e^{-\int_t^U r_s \, ds} \right\} E \left\{ e^{-\int_t^T r_s \, ds} 1\{r_T \leq r^*\} \bigg| F_t \right\} - K E^Q \left\{ e^{-\int_t^T r_s \, ds} 1\{r_T \leq r^*\} \bigg| F_t \right\}$$

$$= E^Q \left\{ e^{-\int_t^U r_s \, ds} 1\{r_T \leq r^*\} \bigg| F_t \right\} - K E^Q \left\{ e^{-\int_t^T r_s \, ds} 1\{r_T \leq r^*\} \bigg| F_t \right\}$$

$$= B_t(U) E^{Q_U} \left\{ 1\{r_T \leq r^*\} \bigg| F_t \right\} - K B_t(T) E^{Q_T} \left\{ 1\{r_T \leq r^*\} \bigg| F_t \right\}$$

$$= B_t(U) Q_U \left\{ 1\{r_T \leq r^*\} \bigg| F_t \right\} - K B_t(T) Q_T \left\{ 1\{r_T \leq r^*\} \bigg| F_t \right\}$$

where $Q_U$ and $Q_T$ are the time-$U$ and time-$T$ forward measures:

$$\frac{dQ_U}{dQ} \bigg| \quad F_t = \frac{e^{-\int_t^U r_s \, ds} B_t(U)}{B_0(U)} \quad \frac{dQ_T}{dQ} \bigg| \quad F_t = \frac{e^{-\int_t^T r_s \, ds} B_t(T)}{B_0(T)}$$

Everything boils down to computing the probability of $\{r_T \leq r^*\}$ given $F_t$ under $Q_U$ and $Q_T$. 