The uncertainty in our economy is modelled as a probability space \((\Omega, \mathcal{F}, P)\) with a filtration \((\mathcal{F}_t)\) representing information available at time \(t\). We assume that there is a \(d\)-dimensional Wiener process \((W_t)\).

**Primary Securities.**

In mathematical finance, we distinguish 2 types of securities: primary securities whose prices are set by supply and demand in the marketplace and derivative securities, which are contracts (usually based on primary securities, but not always) whose price are determined by means of a replicating strategy.

There are \(n+1\) primary securities, \(S^0, \ldots, S^n\).

\(S^0_t\) denote the price at time \(t\) of the risk-free asset \(\frac{dS^0_t}{S^0_t} = r_t \, dt\).

\(S^i_t\), \(i = 1, \ldots, n\) denote price of risky assets, we assume that they follow the following Itô dynamics:

\[
\frac{dS^i_t}{S^i_t} = \mu^i_t \, dt + \sum_{j=1}^d \sigma_{ij} \, dW^j_t
\]

where \(\mu\) and \(\sigma\) satisfy conditions for the above to make sense and \(S^i_t \geq 0\).

**Self-financing trading strategies.**

Given these \(n+1\) primary securities, one can build portfolio (or trading strategies) based on them.

A portfolio is an adapted process \((\psi^i_t)\), \(i = 0, \ldots, n\), representing the number of asset \(i\) to be held at time \(t\).

The value of the portfolio at time \(t\) is \(V_t(\psi) = \sum_{i=0}^n \psi^i_t \cdot S^i_t\).

Among these strategies, we will essentially be only interested in self-financing ones.

A self-financing portfolio is a portfolio whose value evolves so:

\[
dV_t(\psi) = \sum_{i=0}^n \psi^i_t \, dS^i_t
\]

where the \(\psi^i\)'s must satisfy some integrability condition for the above to make sense as an Itô integral.
(a) An arbitrage opportunity is a self-financing portfolio $\Psi$ such that for a given terminal date $T$, $V_0(\Psi) = 0$, $V_T(\Psi) > 0$ and $P\{V_T(\Psi) > 0\} > 0$.

Example: a risk-free portfolio with rate of return $r$.

(b) Assume $d > n$ (i.e., that there are more noise terms than traded securities).

This is not really a restriction since the number of stocks can always be reduced by duplicating some of them as "mutual funds" (i.e., $(t,\omega)$-dependent linear combinations).

The absence of arbitrage essentially implies strong relationships among the different returns: there exists a risk premium vector process $\lambda_t(\omega)$ which is $d$-dimensional (one component for each noise term) such that:

$$\sigma_t^i(\omega) \lambda_t(\omega) = \mu_t(\omega) - \mu^*$$

($\lambda$ is a vector on $n$ ones)

Sketch of the proof: the differential equation giving $V_t$ can be rewritten as follows:

$$dV_t = \Psi_t^o dS_t^o + \sum_{i=1}^{n} \Psi_t^i dS_t^i$$

(since $V_t = \sum_{i=0}^{n} \Psi_t^i S_t^i$)

$$= (V_t - \sum_{i=1}^{n} \Psi_t^i S_t^i) r dt + \sum_{i=1}^{n} \Psi_t^i S_t^i (d\mu_t^i + \sum_{j=1}^{d} \sigma_t^i j dW_t^j)$$

$$= r V_t dt + \sum_{i=1}^{n} \sum_{j=1}^{d} \Psi_t^i S_t^i (d\mu_t^i + \sigma_t^i j dW_t^j).$$

Suppose then exists a vector $\Psi$ such that $\sum_{i=1}^{n} (\Psi^i - r) = 0$ and $\sigma^T (\Psi) = \sum_{i=1}^{n} \sigma^i j \Psi^i = 0$.

such a portfolio would lead to a portfolio with no risk and return $\geq r$.

(i.e., an arbitrage). Therefore: $\sigma^T (\Psi) = 0 \Rightarrow (\Psi - r1)^T (\Psi) = 0$.

that is $\text{Ker}(\sigma^T) \subset (\mu - r1)^T$ i.e., $\mu - r1 \in \text{Im}(\sigma)$.

Given a risk premium vector $\lambda$, dynamics for the stocks rewrite:

$$\frac{dS_t^i}{S_t^i} = \mu_t^i dt + \sum_{j=1}^{d} \sigma_t^i j (dW_t^j + \lambda_t^i dt).$$
Existence of an equivalent local martingale measure.

If the process \( \lambda_t \) satisfies some integrability condition (bounded \( |\lambda_t(u)| < K \) for all \((t,u)\)) or Novikov condition is checked: \( E\left[ \exp \left( \int_0^T |\lambda_t|^2 \, dt \right) \right] < \infty \),

then we can define a new probability \( Q \) on \((\Omega, \mathcal{F}_t, P)\) that is absolutely continuous with respect to \( P \).

\[
\frac{dQ}{dP} \bigg|_{\mathcal{F}_T} = \exp \left( -\int_0^T \sum_{i=1}^d \lambda_t^i \, dW_t^i - \frac{1}{2} \int_0^T |\lambda_t|^2 \, dt \right)
\]

Under \( Q \), \((W_t + \int_0^\cdot \lambda_t \, ds)_t\) is a \( d \)-dimensional Brownian motion \((W_t^Q)_t\).

The \( S \)-dynamics under \( Q \), rewrites:

\[
\frac{dS_t^i}{S_t^i} = r_t \, dt + \sum_{j=1}^d \sigma_{ij} \, dW_t^j.
\]

Define \( \tilde{S}_t^i = \frac{S_t^i}{S_t^j} \) (the discounted stock price). It's formula:

\[
\frac{d\tilde{S}_t^i}{\tilde{S}_t^i} = \sum_{j=1}^d \sigma_{ij}^i \, dW_t^j.
\]

Under \( Q \), \((\tilde{S}_t^i)_t\) are local martingales. (This is why \( Q \) is called a local martingale measure.)

Similarly, the discounted value of a self-financing portfolio is a local martingale under \( Q \):

\[
\tilde{V}_t(Q) = \sum_{i=1}^k \tilde{q}_t^i \, d\tilde{S}_t^i,
\]

where \( \tilde{V}_t(Q) = e^{-\int_0^t \tilde{r}_u \, du} \tilde{V}_0(Q) \).

Up to integrability conditions, the absence of arbitrage implies the existence of a local martingale measure \( Q \). But this is not enough to rule out arbitrage because there are doubling strategies.

Example: \( n = d = 1, \sigma = 1, r = \mu = 0 \). Let \( Q_t = \int_0^t \frac{S_u \, dW_u}{V_T-S} \) \((d\tilde{V}_t = dV_t)\)

\((Q_t)_{0 \leq t \leq T}\) is a martingale \( \left( E\left[ \int_0^T \frac{S_t^2}{V_T} \, dt \right] < \infty \right) \). It is therefore a time-changed Brownian motion: \( B_t^Q = Q_t \) where \( C_t = \langle Q_t \rangle = \int_0^t \frac{S_u^2}{V_T-S} \, du \).

Let \( \tau = \inf \{ t > 0 : B_t = a > 0 \} \). \( Q_{\tau-}^\prime = a \) and \( C_{\tau-}^\prime < T \) a.s. Consider the strategy

\[\Phi_t = \frac{1}{V_T-t} \mathbb{1}_{\{ t < C_{\tau-}^\prime \}}\]

then: \( V_T = \int_0^{\tau-} \Phi_t \, dS_t = a \cdot \tau \leq T \).

Take a \( \alpha \) large as you wish.
Admissible Strategies

It turns out that in order to rule out arbitrage opportunities, we have to restrict the set of self-financing portfolios. This is due to the fact that in continuous time, there are doubling strategies, i.e., there are self-financing strategies \( \tilde{\gamma} \) such that \( V_T(\tilde{\gamma}) = 1 \) and \( V_0(\tilde{\gamma}) = 0 \).

For these strategies, \( \tilde{V}_t(\tilde{\gamma}) \) cannot be a martingale (even though it is a local martingale) under \( Q \). So a simple way to rule them is to look only at self-financing strategies \( \gamma \) such that \( \tilde{V}_t(\gamma) \) is a martingale under \( Q \).

These are called admissible strategies.

There can't be arbitrage opportunities based on admissible strategies: take \( \gamma \) such that \( \tilde{V}_0(\gamma) = 0 \) and \( \tilde{V}_T(\gamma) \geq 0 \), then by the martingale property,

\[
0 \leq E^Q(\tilde{V}_T(\gamma)) \leq \tilde{V}_0(\gamma) = 0.
\]

Therefore, \( \tilde{V}_T(\gamma) = 0 \) \( Q \)-a.s. and since \( P \) and \( Q \) are equivalent, \( V_T(\gamma) = 0 \) \( P \)-a.s.

Attainable claims: For a given maturity date \( T \), we set:

\[
\Phi_T = \{ \tilde{F}_T \text{-measurable } X \text{ such that } \text{there is an admissible trading strategy } \gamma \text{ such that } \tilde{V}_T(\gamma) = X \}\]

\( \Phi_T \) is the set of attainable European contingent claims.

For these claims, pricing and hedging is clear: Hedging is done using the strategy \( \gamma \) and the initial cost of it is:

\[
E^Q[\tilde{V}_T(\gamma)] = \tilde{V}_0(\gamma) = V_0(\gamma)
\]

(\text{Risk-Neutral valuation rule})

The crucial problem is therefore to \underline{kemprove characterize} \( \Phi_T \).

\( S^i_T \) for each \( i \) is of course attainable (take \( \gamma^i_T = 1 \) and \( \gamma^j_T = 0 \) for \( j \neq i \)).

We are more interested in knowing whether \( (S^i_T - K)^+ \) is attainable.

There are 2 ways of dealing with this problem:

(1) Make extra assumptions (most notably that \( \tilde{F}_T \) is generated by Brownian motion alone) and prove that \( \Phi_T = \{ \text{all } \tilde{F}_T \text{-measurable } X \text{ with some integrability condition} \} \) the market is then complete.
(2) In a given model and for a given derivative contingent claim \( c \), one can try to construct a replicating strategy \( V_t(c) \) such that \( V_T(c) = X \). When \( X = f(T, S_T) \) for some function \( f \), and the model is Markovian, we can try to look for \( V_t(c) = g(t, S_t) \) for some function \( g \).

One then shows that \( g \) must satisfy a PDE subject to a terminal condition. If this PDE has a solution and the resulting strategy is admissible, we are done. When models are not Markovian, one can use Malliavin calculus.

In this course, we are mostly concerned with cap, floor, swaption that have payoffs that are functions of zero-coupon bond prices. In the models we will consider, they will be attainable and we will often use the risk valuation rule:

the value at time \( t \) of a contingent claim \( X \) with maturity \( T \) is:

\[
\Pi_t(X) = \mathbb{E}^Q \left\{ e^{-\int_t^T r_s \, ds} X \left| F_t \right. \right\}.
\]
**Definition:** A numéraire is a self-financing portfolio with positive value at any time.

Given a numéraire $Z$, we can express the price of security $S^0, \ldots, S^n$ in terms of this numéraire: \( \frac{S^0}{Z}, \ldots, \frac{S^n}{Z} \).

We already used the value of the bank account $S^0$ as a numéraire to get the discounted asset price: \( \tilde{S}^0 = \frac{S^0}{S^0} = 1, \tilde{S}^1 = \frac{S^1}{S^0}, \ldots, \tilde{S}^n = \frac{S^n}{S^0} \).

We have the following intuitive but important fact:

**Proposition:** Self-financing portfolios remain self-financing when expressed in a different numéraire.

Max mathematically if $\psi$ is a self-financing strategy (i.e., \( dV_t(\psi) = \sum_{i=1}^{n} \psi^i_t \, dS^i_t \)) then

\[
\begin{align*}
\frac{d \left( \frac{V_t(\psi)}{Z_t} \right)}{Z_t} &= \sum_{i=1}^{n} \psi^i_t \, d \left( \frac{S^i_t}{Z_t} \right).
\end{align*}
\]

The quotient Ito rule is

\[
\begin{align*}
\frac{d A_t}{B_t} &= \frac{dA_t}{B_t} - A_t \frac{dB_t}{B_t} - \left( \frac{dA_t}{B_t} - A_t \frac{dB_t}{B_t} \right) \frac{dZ_t}{Z_t}.
\end{align*}
\]

So:

\[
\begin{align*}
\frac{d \left( \frac{V_t(\psi)}{Z_t} \right)}{Z_t} &= \frac{dV_t(\psi)}{Z_t} - V_t(\psi) \frac{dZ_t}{Z_t} - \left( \frac{dV_t(\psi)}{Z_t} - V_t(\psi) \frac{dZ_t}{Z_t} \right) \frac{dZ_t}{Z_t}.
\end{align*}
\]

\[
\begin{align*}
&= \sum_{i=1}^{n} \psi^i_t \left( \frac{dS^i_t}{Z_t} - S^i_t \frac{dZ_t}{Z_t} - \left( \frac{dS^i_t}{Z_t} - S^i_t \frac{dZ_t}{Z_t} \right) \frac{dZ_t}{Z_t} \right)
\end{align*}
\]

\[
\begin{align*}
&= \sum_{i=1}^{n} \psi^i_t \, d \left( \frac{S^i_t}{Z_t} \right).
\end{align*}
\]

We saw that discounted asset price are local martingales under the risk neutral measure $Q$. In other words that asset price expressed in the bank account numéraire are $Q$ local martingale. It turns out that for any numéraire $Z$, there exists a probability measure $\tilde{Q}$ such that asset price expressed in this numéraire are $\tilde{Q}$ local martingale.
Indeed define $Q^Z$ by its Radon-Nikodym derivative with respect to $Q$:

$$\left. \frac{dQ^Z}{dQ} \right|_{\mathcal{F}_t} = \frac{Z_t}{S_t^0 Z_0}.$$ 

Since $Z$ is an admissible selffinancing strategy, $\left. \frac{dQ}{dQ} \right|_{\mathcal{F}_t}$ is a positive martingale with expectation $E \left\{ \left. \frac{dQ}{dQ} \right|_{\mathcal{F}_t} \right\} = \frac{Z_0}{S_0^0 Z_0} = 1$. $Q^Z$ is a well defined probability measure on $(\Omega, \mathcal{F}, P)$ for some $T$.

Let $S^i_t$ be the $i$-th price expressed in numéraire $Z$; and let $T_t \leq T$

$$E \left\{ \left. \frac{S^i_t}{Z_t} \right| \mathcal{F}_t \right\} = \frac{E^Q \left\{ \left. \frac{Z_t}{S_t^0 Z_0} \frac{S^i_t}{Z_t} \right| \mathcal{F}_t \right\}}{E^Q \left\{ \left. \frac{Z_t}{S_t^0 Z_0} \right| \mathcal{F}_t \right\}} = \frac{S^i_t}{Z_t} E^Q \left\{ \left. \frac{Z_t}{S_t^0 Z_0} \right| \mathcal{F}_t \right\} = \frac{S^i_t}{Z_t}.$$ 

Bayes' rule.

Why is this useful? Suppose $X$ is an attainable contingent claim, its arbitrage price at time $t$: $\Pi_t(x)$ is given by

$$\frac{\Pi_t(x)}{S_t^0} = E^Q \left\{ \left. \frac{X}{S_t^0 Z_0} \right| \mathcal{F}_t \right\}.$$ 

Let $Z$ be a numéraire:

$$\frac{\Pi_t(x)}{S_t^0} = Z_t E^Q \left\{ \left. \frac{Z_T}{S_t^0 Z_0} \frac{X}{Z_T} \right| \mathcal{F}_t \right\} = Z_t E^Q \left\{ \left. \frac{Z_T}{S_t^0 Z_0} \right| \mathcal{F}_t \right\} E^Q \left\{ \left. \frac{X}{Z_T} \right| \mathcal{F}_t \right\}$$

Bayes' rule

$$= \frac{Z_t}{S_t^0} E^Q \left\{ \left. \frac{X}{Z_T} \right| \mathcal{F}_t \right\}$$

$\Pi_t(x)$ can also be computed with $Q^Z$:

$$\frac{\Pi_t(x)}{Z_t} = E^Q \left\{ \left. \frac{X}{Z_T} \right| \mathcal{F}_t \right\}.$$ 

Example: $B_t(T)$ is the price of a zero coupon bond maturing at time $T$. It is attainable.

So $B_t(T) = E^Q \left\{ e^{-\int_t^T r_s \, ds} \left| \mathcal{F}_t \right. \right\}$ and we can take it as a numéraire.
Consider the forward contract with delivery at time $T$ on stock $S$. The forward price is by definition the solution to:

$$0 = E^{Q} \left\{ e^{-\int_{t}^{T} \delta_t \, dt} \left( S_T - F_t(T) \right) \mid F_t \right\}$$

(i.e. the price that makes the value of the long position 0 at time $t$).

$$0 = E^{Q} \left\{ \frac{1}{S^2_T} \left( S_T - F_t(T) \right) \mid F_t \right\}$$

which rewrites under $Q^T$, the probability measure associated with the numeraire $B_t(T)$:

$$0 = E^{Q^T} \left\{ \frac{1}{B_t(T)} \left( S_T - F_t(T) \right) \mid F_t \right\} = E^{Q^T} \left\{ S_T \mid F_t \right\} - F_t(T).$$

Since $B_t(T) = 1$ and $F_t(T)$ is $F_t$-measurable.

Therefore:

$$F_t(T) = E^{Q^T} \left\{ S_T \mid F_t \right\}$$

Another example: Consider the 2-dimensional Black-Scholes model:

$$\frac{dS_T^1}{S_T^1} = r \, dt + \sigma^1 \, dW_t^1 \quad \text{and} \quad \frac{dS_T^2}{S_T^2} = r \, dt + \sigma^2 \, dW_t^2,$$

where $E^{W_t^1 W_t^2} = p(t)$.

The exchange option pays $(S_T^2 - S_T^1)^+$ at time $T$.

Its price is:

$$p_t = E^{Q} \left\{ e^{-r(T-t)} \left( S_T^2 - S_T^1 \right)^+ \mid F_t \right\}$$

under $Q^{S^1}$:

$$p_t = S_t^1 \ E^{Q^{S^1}} \left\{ \frac{1}{S_T^1} \left( S_T^2 - S_T^1 \right)^+ \mid F_t \right\}$$

$$= S_t^1 \ E^{Q^{S^1}} \left\{ \left( \frac{S_T^2}{S_T^1} - 1 \right)^+ \mid F_t \right\}$$

looks like Black-Scholes Call option.

We need to know the distribution of $\frac{d(S_T^2 / S_T^1)}{S_T^2 / S_T^1}$ under $Q^{S^1}$.

Under $Q$:

$$\frac{d(S_T^2 / S_T^1)}{S_T^2 / S_T^1} = \frac{dS_T^2}{S_T^2} - \frac{dS_T^1}{S_T^1} = \left\langle \frac{dS_T^2}{S_T^2} - \frac{dS_T^1}{S_T^1}, \frac{dS_T^1}{S_T^1} \right\rangle$$

$$= \sigma^2 \, dW_t^2 - \sigma^1 \, dW_t^1 - \left\langle \sigma^2 dW_t^2 - \sigma^1 dW_t^1, \sigma^1 dW_t^1 \right\rangle$$

$$= \sigma^2 \, dW_t^2 - \sigma^1 \, dW_t^1 - (\sigma^1 p - (\sigma^1)^2) \, dt.$$
a) Under $Q^S$: $S_t^2/S_t^1$ is a local martingale.

The quadratic variations do not change when we change probability.

\[ \langle d(S_t^2/S_t^1) \rangle = \langle \sigma^2 dw_t^2 - \sigma^1 dw_t^1 \rangle = \left( \sigma^2 \right)^2 - 2\rho \sigma^1 \sigma^2 + (\sigma^1)^2 \ dt. \]

So under $Q^S$: \[ \frac{dS_t^2}{S_t^2/S_t^1} = \bar{\sigma} \tilde{W}_t \]

where \[ \bar{\sigma} = \sqrt{(\sigma^2)^2 - 2\rho \sigma^1 \sigma^2 + (\sigma^1)^2} \]

$\tilde{W}$ Brownian motion under $Q^S$.

Therefore \[ p_t = S_t^1 \left[ \frac{S_t^2}{S_t^1} \Phi(d_1) - \Phi(d_2) \right] = S_t^1 \Phi(d_1) - S_t^1 \Phi(d_2). \]

\[ d_1 = \frac{\ln \left( S_t^2/S_t^1 \right)}{\bar{\sigma} \sqrt{T-t}} + \frac{1}{2} \bar{\sigma} \sqrt{T-t} \]

\[ d_2 = d_1 - \bar{\sigma} \sqrt{T-t}. \]

b) Or we compute \[ \frac{dQ^S}{dQ} \bigg|_{\mathcal{F}_t} = \frac{S_t^1}{S_0^1} e^{-r t} = e^{-\frac{\sigma_t W_t^1 - \sigma_t^2 t}{2} + \int_0^t \int_0^t \int_0^t (\sigma^1)^2 do} \]

under $Q^S$, $W_t^1 - \sigma_t t$ is Brownian motion

\[ W_t^2 = \rho W_t^1 + \sqrt{1-\rho^2} \tilde{W}_t^1 \]

where $\tilde{W}_t^1$ is Brownian motion independent of $W_t^1$ under $Q$.

under $Q^S$, $W_t^2 - \rho \sigma_t t$ is Brownian motion. Because $\tilde{W}_t^1$ is also Brownian motion, $\tilde{W}_t^1 W_t^1$ under $Q^S$. 

So \[ \frac{d(S_t^2/S_t^1)}{S_t^2/S_t^1} = \sigma^2 dw^2_t - \sigma^1 dw^1_t - \left( \sigma^1 \sigma^2 - (\sigma^1)^2 \right) dt \]

\[ = \sigma^2 \left( dw^2_t - \rho \sigma t dt \right) - \sigma^1 \left( dw^1_t - \sigma t dt \right) \]

BM under $Q^S$  BM under $Q^S$