Futures contracts

A futures contract looks like a forward contract.

In a futures contract two parties agree that one will pay the other for an asset at time \( T \) at a price agreed on today \( f_0(T) \). (Today is time 0).

\( f_0(T) \) is chosen such that the contract is costless.

Of course at time \( T \): \( f_T(T) = S_T \) (for otherwise, there would be an arbitrage).

Contrary to forward contracts, both of the payments are made between initiation and maturity. Namely, at the end of each trading day the long position receives \( f_t(T) - f_{t-1}(T) \) from the short position.

(If this is negative, the amount is paid). At maturity, the long position only pays \( f_T(T) = S_T \) since the difference \( f_0(T) - S_T \) has been paid already:

\[
    f_0(T) = S_T = \sum_{t=1}^{T} f_t(T) - f_{t-1}(T)
\]

How to find \( f_0(T) \)? Seems more complicated since all \( f_t(T) \) for \( 0 \leq t \leq T \) are playing a role.

Consider the following strategy: at each time \( t \) enter the long position in \( e^{rt} \) futures contract and close the position at time \( t + dt \).

We only receive \( f_{t+dt}(T) - f_t(T) \) since entering and closing such position is costless. This amount is then put in bank account until time \( T \).

At time \( T \), this strategy leads to

\[
    \int_0^T e^{r(T-t)} e^{rt} df_t(T)
\]

Which is

\[
    e^{rt} \int_0^T df_t(T) = e^{rt} (f_T(T) - f_0(T)) = e^{rT} (S_T - f_0(T))
\]

Moreover, this strategy is costless at time 0. It is equivalent to \( e^{rT} \) the long position in a forward contract on \( S \) with maturity \( T \):

\[
    f_0(T) = F_0(T)
\]

Forward price = Futures price. But this is not true when interest rates are stochastic.
Example of a Swap:

Company A is borrowing fixed for 5 years at 5\(\frac{1}{2}\)% but could borrow at LIBOR + 1%.  
Company B is borrowing floating at LIBOR + 1% but could borrow fixed for 5 years at 6\(\frac{1}{2}\)%. 

By agreeing to swap streams of cashflows both companies could be better off and a mediating institution would also make money.  
Company A pays LIBOR to the intermediary in exchange for fixed at 5\(\frac{3}{16}\)% (receiver swap)  
Company B pays fixed to the intermediary at 5\(\frac{5}{16}\)% in exchange for LIBOR (payer swap)  
Company A is now paying LIBOR + 5\(\frac{5}{16}\)% instead of LIBOR + 6\(\frac{1}{2}\)%  
Company B is now paying fixed at 6\(\frac{5}{16}\)% instead of 6\(\frac{1}{2}\)%  
The intermediary receives fixed at 1/8%.

Everyone seems better off but there is an implicit credit risk this is why company B had higher borrowing rates in the first place. This risk has been partly taken up by the intermediary in return for the money it makes on the spread.

A remark about terminology

The party paying fixed and receiving floating is said to hold a payer swap.

The party paying floating and receiving fixed is said to hold a receiver swap.

Definition of \(f_t(T)\), the instantaneous, continuously compounded forward rate:

\[
f_t(T) = \lim_{h \to 0} \frac{R_t(T, T+h) - R_t(T, T)}{h} \bigg|_{T=t}
\]

Definition of \(r_t\), the short rate:  
\(r_t = f_t(t)\).
Bond Market

There are OTC bond options, i.e., options to buy (or to sell) bonds at a
certain date for a certain price. Since there are a lot of different types of bonds,
the market is rather illiquid.

More liquid are T-Bond futures options, i.e., options to enter a futures contract
on a T-Bond. This protects against too high (or too low) futures price at
maturity of the option.

Money Market

Caps: A caplet with reset date \( T \) and settlement \( T + \delta \) pays the holder the
difference between the LIBOR rate and the strike rate \( k \)
Cashflow at time \( T + \delta \) : \( \delta (L_T (T + \delta) - k)^+ \)
A cap is a strip of caplets, take future dates \( T_0 < T_1 < \ldots < T_n \) \((T_i - T_{i-1} = \delta)\)
and cap rate \( k \).

At each time \( T_i \), \( \ldots, T_n \), the holder of the cap receives
\( \delta (L_{T_{i-1}} (T_i) - k)^+ \) at time \( T_i \).

Thus a cap guarantees a maximum rate for each period \( (T_{i-1}, T_i) \).
It will be of interest for a company with a loan with payment indexed in
LIBOR.

If we write \( \text{Caplet}_i (t) \) for the pay at time \( t \) of the \( i \)-th caplet,
the pay at time \( t \) of the cap is:
\[
\text{Cap} (t) = \sum_{i=1}^{n} \text{Caplet}_i (t)
\]

Floors: A floorlet pays \( \delta (k - L_T (T + \delta))^+ \) at time \( T + \delta \).
It protects against low rates.
A floor is a series of floorlets, each paying \( \delta (k - L_{T_{i-1}} (T_i))^+ \) at time \( T_i \)
\[
\text{Floor} (t) = \sum_{i=1}^{n} \text{Floorlet}_i (t)
\]

Floors and caps are related to swaps in the following way:
\[
\text{Cap} (t) - \text{Floor} (t) = \text{TP}_t (t)
\]
where \( \text{TP}_t (t) \) is the value at time \( t \) of a payer swap with swap rate \( k \) and swap payment date
at the cap and the floor. (this is the spread of the call and put options).
For that reason, the cap or the floor is said to be at-the-money if
\[ s = R_{\text{swap}}(t) = \frac{B_t(T_n) - B_t(T_0)}{\delta \sum_{i=1}^{n} B_t(T_i)} \]

Market convention:

The market quotes caplets, floorlets, caps and floors in terms of their implied volatilities to be plugged into Black's formula:

\[ \text{Caplet}_t(T_i) = \delta B_t(T_i) \left[ L_t(T_i, S_i) \Phi(d_1) - \delta \Phi(d_2) \right] \]

where \[ d_{1, 2} = \frac{\ln \left( \frac{B_t(T_i, S_i)}{K} \right) + \frac{1}{2} \sigma^2 T_i}{\sigma \sqrt{T_i}} \]

When quoting a cap, all implied volatilities for each caplet is taken to be the same \( \sigma \).

(cap implied volatility)

In case of floorlets:

\[ \text{Floorlet}_t(T_i) = \delta B_t(T_i) \left[ \delta \Phi(-d_2) - L_t(T_i, S_i) \Phi(-d_1) \right] \]

Swaptions:

A European payer swaption with strike \( K \) is an option giving the right to enter a payer swap with strike \( K \) at a given future date called the swaption's maturity.

A European receiver swaption is defined analogously.

Usually, the swaption's maturity coincides with the first reset date of the underlying swap. (\( T_0 \) in the previous notation for swap.) \( T_n - T_0 \) is called the tenor of the swaption.

A European payer swaption protects against high swap rates at a given future date and will be of interest to investors planning to enter a payer swap in the future.

The payoff of a payer swaption at maturity \( T_0 \) is

\[ N \left( \sum_{i=0}^{n} B_{T_0}(T_i) - (1 + \delta K) \sum_{i=1}^{n} B_{T_0}(T_i) \right)^+ = N \left( B_{T_0}(T_n) - \delta K \sum_{i=1}^{n} B_{T_0}(T_i) \right)^+ = \delta \sum_{i=1}^{n} B_{T_0}(T_i) (R_{T_0}^{\text{swap}}(T_n) - K)^+ \]

Contrary to the case of caps and floors, the payoff cannot be written as a sum of more elementary payoffs. Swaptions are more complex because of correlations.
and the receiver swap: \( N \delta (K - k_{\text{swap}}) \sum_{i=1}^{n} \beta(T_i) \).

From this, it is also clear that

\[ \text{payer swap}_{t} (K) - \text{receiver swap}_{t} (K) = \text{payer swap}_{t} (K). \]

**Market convention:**

The market quotes payer and receiver swaps in terms of their Black-Scholes implied volatilities,

\[
\begin{align*}
\text{payer swap}_{t} & = N \delta \left( k_{\text{swap}} (t) \Phi(d_1) - K \Phi(d_2) \right) \sum_{i=1}^{n} \beta_i(T_i) \\
\text{receiver swap}_{t} & = N \delta \left( K \Phi(-d_2) - k_{\text{swap}} (t) \Phi(-d_1) \right) \sum_{i=1}^{n} \beta_i(T_i) \\
\text{with} \ d_{1,2} & = \frac{\ln \left( \frac{k_{\text{swap}} (t)}{K} \right) + \frac{1}{2} \sigma_{T}^{2} (T_0 - t)}{\sigma_{T} \sqrt{T_0 - t}}
\end{align*}
\]

A zero swap refers to a swap maturing in \( n \) years and whose underlying swap has maturity \( y \) years after the maturity of the swap. Thus, leads to a matrix swap's volatility.

![ATM Cap volatility](image-url)
Most important rates.

Today is time \( t \), \( t < T_1 < T_2 \) and \( t < T \).

### Spot rates

- Continuously compounded spot rate \( B_t(T) \): \[ B_t(T) = \exp(-R_t(T-t)) \]
- Simply compounded spot rate \( L_t(T) \): \[ B_t(T) = \frac{1}{1+L_t(T)(T-t)} \]

### Instantaneous short rate \( f_t \):

\[ f_t(T) = \lim_{T+\Delta T \to 0} \frac{B_t(T+\Delta T) - B_t(T)}{\Delta T} \]

### Forward rates

- Continuously compounded forward rate \( R_{T_1,T_2}(t) \): \[ R_{T_1,T_2}(t) = \frac{B_t(T_1)}{B_t(T_2)} \exp(-R_t(T_1,T_2) (T_2-T_1)) \]
- Simply compounded forward rate \( L_{T_1,T_2}(t) \): \[ B_t(T_1,T_2) = \frac{1}{1+L_{T_1,T_2}(T_1,T_2)(T_2-T_1)} \]

### Bank account:

It is the value of 1 dollar that is invested everyday in bonds maturing the day after (i.e., overnight LIBOR contracts).

Its value at time \( t \): \[ B_t = \exp\left( \int_0^t r_s \ ds \right) \]

and solves the ordinary differential equation: \( \frac{dB_t}{B_t} = f_t dt \)

At this point, we can ask ourselves why we need fixed income models and why we don't just use the Black-Scholes theory for stocks?

There are different reasons for that:

1. \( r \) is assumed constant or deterministic in Black-Scholes. assumption we can't make here!

2. Options are often written on rates and then are not traded assets per se. so we can't directly use them for replication.

3. Some options (like Swaption) have payoffs that depend on different rates. we need to model them in a consistent manner. Unlike two different stocks of different companies, two zero coupon bond of different but close maturities have strong relation.