Bond market

The fixed income markets are usually separated into bond market (where debt securities of government are traded) and money market (where contracts are traded among banks).

The most important security in fixed income markets is a zero coupon bond. It is a security that pays 1 dollar at a specified date in the future $T$. Its price at time $t \leq T$ is denoted $B_t(T)$.

Of course, $B_t(T) = 1$ for all $T$ (for otherwise, there is an obvious arbitrage) and usually $B_t(T) \leq 1$ (a dollar later is worth less than a dollar today).

Note the difference:

Comparing bond prices with different maturity is not straightforward, so we introduce the following definitions.

a) The continuously compounded yield rate $\gamma_t(T)$ is such that

$$B_t(T) = \exp(-\gamma_t(T) \cdot (T-t))$$

b) The simply compounded yield rate, more commonly called simple rate $\lambda_t(T)$

$$B_t(T) = \frac{1}{1 + \lambda_t(T) \cdot (T-t)}$$

(aside: the notation $\gamma_t(T)$ will become clear later)

US Bond market: Debt securities issued by the US Treasury are divided into

- **T-Bills**: Zero coupon bond maturity $\leq 1$ year
- **T-Notes**: Semi-annual coupon bonds $2\text{yr} \leq \text{maturity} \leq 10\text{yr}$
- **T-Bonds**: Semi-annual coupon bonds maturity $> 10\text{yr}$

Given a zero coupon bond curve at time $t$ (i.e., $T \mapsto B_t(T)$) the price of a coupon bond is coupon rate $c_i$ and payment dates $T_i, \cdots, T_m$ and nominal amount $N$ is

$$N \left( \sum_{i=1}^{m} \frac{1}{2} c_i B_t(T_i) + \left( 1 + \frac{1}{2} c_i \right) B_t(T_m) \right)$$

Conversely, given prices of US Treasury securities and an interpolation procedure, we can produce a zero coupon curve; this is called STRIPPING.
Without loss of generality we can assume in the formula above that \( N = 1 \) and we can replace \( \frac{1}{2} B(t) \) by \( 1 + c_n/2 \) so that the price at time \( t \) is

\[
    p(t) = \sum_{i=1}^{n} c_i B(t_i) \quad \text{when} \quad t \leq T_1
\]

and between \( (T_1, T_2] \),

\[
    p(t) = \sum_{i=2}^{n} c_i B(t_i) \quad \text{if} \quad T_1 < t \leq T_2
\]

Due to the payment of the coupon \( t \mapsto p(t) \) is discontinuous (like stock and dividends).

Although this is the actual price of the bond, the market quotes another price that has no discontinuities when coupons are paid.

Define the accrued interest at time \( t \in [T_{i-1}, T_i) \) to be

\[
    AI(z, t) = \frac{t - T_{i-1}}{T_i - T_{i-1}} c_i
\]

the quoted price, also called clean price is

\[
    p_{\text{clean}}(t) = p(t) - AI(z, t) \quad \text{in} \quad [T_{i-1}, T_i)
\]

the cash price \( p(t) \) is also called the dirty price.

Yield-to-Maturity

At time \( T_i \), the (annual) yield-to-maturity \( \frac{c_i}{T_i} \) is defined as

\[
    \frac{c_i}{T_i} = \sum_{j=i+1}^{n} \frac{c_j/2}{(1 + \frac{c_j}{2})^{j-i}}
\]

At \( t \in [T_{i-1}, T_i) \),

\[
    p(t) = \sum_{j=i+1}^{n} \frac{c_j/2}{(1 + \frac{c_j}{2})^{j-i-1+t}} \quad \text{with} \quad T = \frac{T_{i-1} - t}{T_i - T_{i-1}}
\]

Since coupons are paid semi-annually, \( T_{i+1} - T_i = \frac{1}{2} \), and \( c_i \) is the Annualized Coupon.

One can also define the continuously compounded yield to maturity \( y_c \) at time \( t \in [T_i, T_{i+1}) \) as

\[
    p(t) = \sum_{j=i+1}^{n} c_j e^{-y_c (T_j - t)}
\]

A word about day-count conventions. Time is measured in years. There are three different ways of counting the time between \( T_1 \) and \( T_2 \):

- Actual / 365: \((\text{actual number of days between } T_1 \text{ and } T_2) / 365)
- Actual / 360: \((\text{actual number of days between } T_1 \text{ and } T_2) / 360)
- 30 / 360: \(\frac{\min(d_2, 30) + (30-d_1)^+}{360} + \frac{(m_2 - m_1 + 1)^+}{12} + \frac{y_2 - y_1}{2}

Months count 30 days, years 360 days. For instance, in that convention July 4, 2002 - July 4, 2003 counts 2.5 years.
The main contracts traded among large financial institutions are: LIBOR, FRA, futures and swaps.

1) **LIBOR (London Interbank Offered Rate)**: these are rates at which banks borrow and lend. They are simple rates denoted \( L_t^e(T) \) where \( T \) is the maturity of the loan and \( t \) is the current date. \( L_t^e(T) \) is known at time \( t \) when the loan is initiated.

\[
L_t^e(T) = \frac{1}{1 + (T-t) L_t^e(T)}
\]

2) **FRA (Forward Rate Agreement)**: A FRA is an agreement today (time \( t \)) for a loan between \( T_1 \) and \( T_2 \). It is very much like a forward contract.

\[
L_t^e(T_1, T_2) = \frac{1}{1 + (T_2-T_1) L_t^e(T_1, T_2)}
\]

The FRA is denoted by \( L_t^e(T_1, T_2) \).

From today, it is equivalent to \(- (1 + (T_2-T_1) L_t^e(T_1, T_2)) B_t^e(T_2) + B_t^e(T_1)\).

\[
L_t^e(T_1, T_2) = \frac{1}{T_2 - T_1} \left( \frac{B_t^e(T_1)}{B_t^e(T_2)} - 1 \right)
\]

Of course, \( L^{-1}_{T_1}(T_1, T_2) = L_{T_1}(T_2) \).

Note that \( \frac{B_t^e(T_2)}{B_t^e(T_2)} \) is the forward price of the \( T_2 \)-bond for delivery at time \( T_1 \).

Usually denoted \( B_t^e(T_1, T_2) \) and \( B_t^e(T_1, T_2) = \frac{1}{1 + (T_2-T_1) L_t^e(T_1, T_2)} \).

\( L_t^e(T_1, T_2) \) is also called forward LIBOR, or simply compounded forward rate, or single forward rate.

Similarly, define the continuously compounded forward rate:

\[
B_t^e(T_1, T_2) = \exp - \int_{T_1}^{T_2} \frac{1}{t} \, dt.
\]
3) Eurodollar futures.

Eurodollars are dollars deposited in a US or foreign bank outside the US. The Eurodollar future contract is tied to LIBOR. Like a FRA, it is designed to protect its owner from fluctuations in the LIBOR market.

Delivery dates (i.e., the starting dates of the loan) are standardized, they are on the 3rd Wednesday of March, June, September, and December up to 10 years in the future. The duration of the loan is usually 3-months. At time $t \leq T$, the futures LIBOR is denoted by $L^*_t(T, T+\delta) \delta = 1/4$. $T$ is delivery date. The market quotes it as $100 \left( 1 - L^*_t(T, T+\delta) \right)$. The only difference with a FRA is that it is marked-to-market every day like any other future contract. The formula used for marking to market is

$$1 - \delta L^*_t(T, T+\delta) \text{ millions of dollars.}$$

When $\delta = \frac{1}{4}$, it means that when the future rate drops by 1bp = $10^{-4}$ the long position pays 25 dollars to the short position. At maturity, a contract is worth $1 - \delta L_T(T+\delta)$ by definition. It is not exactly what we would have expected, namely, $B_T(T+\delta) = 1 - \delta B_T(T+\delta) L_T(T+\delta)$.

4) Swap.

An interest rate swap is a contract between two parties where one party exchanges a fixed rate against a floating rate from the other party.

Let $t \leq T_0 < \ldots < T_n$, where $t$ is today's date and $T_0, \ldots, T_n$ are dates at which payments occur. $T_n$ is called the maturity of the swap. Usually $T_i - T_{i-1} = \delta = 1$ year or $\frac{1}{2}$ year. When $t < T_0$, we speak about a forward swap. If we denote by $K$ the fixed rate, and $N$ the notional amount, at each time $T_i$ ($i > 1$) the party called payer swap, pays fixed $KN$.

receive floating $\delta L_{T_{i-1}}(T_i) \frac{N}{T_i}$.

LIBOR observed at the beginning of period $(T_{i-1}, T_i)$.

the other party is called the receiver swap.
Take $N = 1$ for simplicity.

Floating leg:

$$
\delta L_{T_0}(T_1) \quad \delta L_{T_0}(T_2) \quad \delta L_{T_0}(T_3) \quad \delta L_{T_0}(T_4) \quad \delta L_{T_0}(T_5) \quad \delta L_{T_0}(T_6) \quad \delta L_{T_0}(T_7)
$$

Fixed leg:

$$
\delta K \quad \delta K \quad \delta K \quad \delta K \quad \delta K \quad \delta K
$$

$$
H + \delta L_{T_0}(T_1) \quad H + \delta L_{T_0}(T_2) \quad H + \delta L_{T_0}(T_3) \quad H + \delta L_{T_0}(T_4) \quad 1 + \delta L_{T_0}(T_5)
$$

$$
H + \delta K \quad H + \delta K \quad H + \delta K \quad H + \delta K \quad 1 + \delta K
$$

K is set such that the value at initiation (time $t$) is 0 it is called the forward swap rate $R_{\text{swap}}^{\text{forward}}(T_0, T_n)$.

The present value of the cash flows is $B_c(T_0) - B_c(T_n) - \delta K \sum_{i=1}^{n} B_t(T_i)$.

So that

$$
R_{\text{swap}}^{\text{forward}}(T_0, T_n) = \frac{B_c(T_0) - B_c(T_n)}{\delta \sum_{i=1}^{n} B_t(T_i)}
$$