1.1. Problem 9

(a) \((a+b\sqrt{2})+(c+d\sqrt{2})\) is defined to be \((a+c)+(b+d)\sqrt{2}\). If \(a, b, c, d \in \mathbb{Q}\), then \(a+c, b+d\) are also in \(\mathbb{Q}\). Therefore + is a binary operation on \(G\).

To prove \((G,+)\) is a group, we need to check three conditions.

(i) Associativity.

Associativity means: for any \(a, b, c, d, e, f \in \mathbb{Q}\),

\[
[(a + b\sqrt{2}) + (c + d\sqrt{2})] + (e + f\sqrt{2}) \quad \text{(expand first two first)}
\]
\[
= (a + b\sqrt{2}) + [(c + d\sqrt{2}) + (e + f\sqrt{2})] \quad \text{(expand last two first)}
\]
is true.

Reason:

\[
[(a + b\sqrt{2}) + (c + d\sqrt{2})] + (e + f\sqrt{2})
\]
\[
=[(a + c) + (b + d)\sqrt{2}] + (e + f\sqrt{2})
\]
\[
=[(a + c + e) + (b + d + f)\sqrt{2}]
\]
\[
=(a + b\sqrt{2}) + [(c + e) + (d + f)\sqrt{2}]
\]
\[
=(a + b\sqrt{2}) + [(c + d\sqrt{2}) + (e + f\sqrt{2})]
\]

(ii) Existence of identity

It’s clear \(0 + (a + b\sqrt{2}) = (a + b\sqrt{2}) + 0 = (a + b\sqrt{2})\). So 0 is the identity element in \(G\).
(iii) \( 0 = (a + b\sqrt{2}) + (-a - b\sqrt{2}) = (-a - b\sqrt{2}) + (a + b\sqrt{2}) \). So \(-a - b\sqrt{2}\) is the inverse of \(a + b\sqrt{2}\).

(b) \((a+b\sqrt{2})(c+d\sqrt{2})\) is defined to be \((ac+2bd)+(bc+ad)\sqrt{2}\). If \(a, b, c, d\in Q\), then \(ac + 2bd, bc + ad\) are also in \(Q\). Therefore multiplication is a binary operation on \(G\). To prove \((G, \cdot)\) is a group, we need to check three conditions.

(i) Associativity.

Associativity means: for any \(a, b, c, d, e, f \in Q\)

\[
[(a + b\sqrt{2})(c + d\sqrt{2})](e + f\sqrt{2})(\text{expand first two first}) \\
= (a + b\sqrt{2})[(c + d\sqrt{2})(e + f\sqrt{2})](\text{expand last two first})
\]

is true.

Reason:

\[
[(a + b\sqrt{2})(c + d\sqrt{2})](e + f\sqrt{2}) \\
= [(ac + 2bd) + (ad + be)\sqrt{2}](e + f\sqrt{2}) \\
= [(ace + 2bde + 2adf + 2bcf) + (acf + 2bdf + ade + bce)\sqrt{2}] \\
= (a + b\sqrt{2}) + [(ce + 2df) + (de + cf)\sqrt{2}] \\
= (a + b\sqrt{2})(c + d\sqrt{2})(e + f\sqrt{2})
\]

(ii) Existence of identity

It’s clear \(1 \cdot (a + b\sqrt{2}) = (a + b\sqrt{2}) \cdot 1 = (a + b\sqrt{2})\). So 1 is the identity element in \((G, \cdot)\).

(iii) If \(a, b \in Q\), \(\frac{1}{a+b\sqrt{2}} = \frac{a-b\sqrt{2}}{(a+b\sqrt{2})(a-b\sqrt{2})} = \frac{a-b\sqrt{2}}{a^2-2b^2}\) is in \(G\). This is the inverse of \(a + b\sqrt{2}\).

1.1. Problem 12

\(\bar{1}\) has order 1.

\[\bar{1}^2 = 1, \text{ so } \bar{1} \text{ has order } 2.\]

\[\bar{5}^2 = 25 = \bar{1}, \text{ so } \bar{5} \text{ has order } 2.\]

\[\bar{7}^2 = 49 = \bar{1}, \text{ so } \bar{7} \text{ has order } 2.\]

\[\bar{7}^2 = 49 = \bar{1}, \text{ so } \bar{7} \text{ has order } 2.\]
13 = 1 has order 1.

1.1. Problem 15

Because

\[(a_1 a_2 \ldots a_n)(a_n^{-1} a_{n-1}^{-1} \ldots a_1^{-1}) = (a_1 a_2 \ldots a_{n-1})(a_{n-1}^{-1} a_{n-2} \ldots a_1^{-1})\]
\[= \ldots = e\]

\[(a_n^{-1} a_{n-1}^{-1} \ldots a_1^{-1})(a_1 a_2 \ldots a_n) = (a_n^{-1} a_{n-1}^{-1} \ldots a_2^{-1})(a_2 a_3 \ldots a_n)\]
\[= \ldots = e\]

This implies \(a_n^{-1} a_{n-1}^{-1} \ldots a_1^{-1}\) is the inverse of \(a_1 a_2 \ldots a_n\).

Note: You can also use induction on \(n\) to prove it.

1.2. Problem 6

From assumption \(x, y\) are elements of order 2. \(x^2 = e, y^2 = e\). Then we have \(x = x^{-1}, y = y^{-1}\).

\[tx = (xy)x\]

\[xt^{-1} = x(y^{-1}x^{-1}) = x(yx) = tx\]

1.2. Problem 15

\[\{1 \mid n \cdot 1 = 0\}\]

For \(n \cdot 1\) mean \(1 + 1 + \ldots + 1\) \(n\) times.