Goal: understand \( \lim_{g \to \infty} \mathbb{H}_\ast (\text{Diff}^+\Sigma_g) \),
prove the Mumford conjecture:
\[
\lim_{g \to \infty} \mathbb{H}_\ast (\text{Diff}^+\Sigma_g; \mathbb{Q}) = \mathbb{Q}[e_1, e_2, e_3, \ldots]
\]

Outline:
- \( \text{Diff}^+\Sigma_g \) = subsets of \( \mathbb{R}^\infty \) diffeomorphic to \( \Sigma_g \)
- \( \underleftarrow{\text{Diff}^+\Sigma_g} \) = subsets of \( \mathbb{R}^\infty \) diffeomorphic to a closed surface
- \( S(0,N) \) = subsets of \( \mathbb{R}^N \) diffeomorphic to a closed surface

\( S(N,N) \) = properly embedded 2-dimensional manifolds in \( \mathbb{R}^N \),

topologized so manifolds can disappear at infinity

(not necessarily compact, connected, finite type, etc.)

\( S(k,N) \) = subspace of \( S(N,N) \) consisting of properly embedded
2-dimensional manifolds contained in \( \mathbb{R}^k \times (0,1)^{N-k} \)

\( S(1,N) \) = properly embedded 2-dimensional manifolds contained in \( \mathbb{R} \times (0,1)^{N-1} \)

\( S(0,N) \) = 2-dimensional closed manifolds in \( (0,1)^N \)

Problem: operation in \( S(0,N) \) is disjoint union:

We will redefine, get \( \Sigma(0,N) \) s.t.
operation is connected sum and \( \lim_{N \to \infty} \Sigma(0,N) = \underleftarrow{\text{Diff}^+\Sigma_g} \)

Relaxation principle:
- \( BS(0,N) = S(1,N) \)
- \( BSC(1,N) = S(2,N) \)
- \( BS(0,N) = S(N,N) \)

Zooming: \( S(N,N) \cong \text{Aff}^+(\mathbb{R}^N) \)

Combining these, we have
\[
\mathbb{H}_\ast (\Sigma(0,N); \mathbb{Q}) \cong \mathbb{H}_\ast (\Omega^\infty \text{Aff}^+(\mathbb{R}^N))
\]

Lecture 8: why is this the cohomology?
how do these classes translate to characteristic classes of surface bundles?
\( \Pi_0(S(k,N)) \) is a group for \( k > 0 \)

Let \( d = 2 \), so that \( S(k,N) \) is the space of \( d \)-manifolds in \( \mathbb{R}^N \),
allowed to go to infinity in \( k \) directions.

We will give an argument for \( d = 2 \) which works verbatim for all \( d \).
Assume \( N > d \). Then:

for \( 0 < k \leq d \), \( \Pi_0(S(k,N)) = \) the group \( \Omega^{\infty}_{d-k,N-k} \) of cobordism classes
of oriented smooth \( (d-k) \)-manifolds in \( \mathbb{R}^{n-k} \)
(with cobordisms embedded in \( \mathbb{R}^{n-k} \times \mathbb{I} \))

for \( d < k \), \( \Pi_0(S(k,N)) = 0 \).

(So for \( d = 2 \), \( \Pi_0(S(1,N)) = 0 \), \( \Pi_0(S(2,N)) = \mathbb{Z} \), \( \Pi_0(S(k,N)) = 0 \) for \( k > 2 \))

The map \( S(k,N) \rightarrow \Omega^{\infty}_{d-k,N-k} \) is given by intersecting with a generic \( (0,1)^{N-k} \):

Well-defined? If we intersect with 
a different \( (0,1)^{N-k} \), the slab
between them is a cobordism
embedded in \( (0,1)^{N-k} \times \mathbb{I} \).

Surjective? for any \( [M] \in \Omega^{\infty}_{d-k,N-k} \), we can take \( \mathbb{R}^k \times M \in S(k,N) \):

Injective? First, note that anything in \( S(k,N) \) can be homotoped to be of the form \( \mathbb{R}^k \times M \):

So any failure of injectivity is because \( M \) and \( N \) are cobordant, but \( \mathbb{R}^k \times M \) and \( \mathbb{R}^k \times N \)
lie in different components of \( S(k,N) \).

But, if \( M \) and \( N \) are cobordant, we can homotope \( \mathbb{R}^k \times M \) to \( \mathbb{R}^k \times N \) in \( S(k,N) \)
by zipping.
$BS(k,N) = S(k+1,N)$ for $k \geq 0$

Caveats:
- We make $S(k,N)$ into a monoid by taking surfaces in $\mathbb{R}^k \times (0,\alpha) \times (0,1)$ for $\alpha > 0$, with operation given by juxtaposition in the $k+1$st coordinate.
- Since $BS(k,N)$, we'd better take only one component $S(k+1,N)_0$; this doesn't affect the outline, since $BS(k,N) = S(k,N)_0$ gives $S(k,N) = \Omega S(k+1,N)_0 = \Omega S(k+1,N)$.

\[ S(k+1,N)_0 \cong S_{\text{walls}}(k+1,N) \cong \tilde{\cong} \rightarrow BS(k,N) \]

A point in $S_{\text{walls}}(k+1,N)$ is:
- A surface $S \in S(k+1,N)$,
- $n+1 \geq 1$ walls $W_i$ disjoint from $S$
- Weights $t_i \geq 0$ with $t_0 + \ldots + t_n = 1$

The map $S_{\text{walls}}(k+1,N) \cong BS(k,N)$ is defined just as before:
- After forgetting what's outside the walls, the walls cut into slabs giving elements
  \[
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  e S(k,N),
  \]
  and together with the weights $t_i$ we have coordinates in $S(k,N) \times S(k,N) \times \Delta^2 \subset BS(k,N)$.

Since fibers are contractible, this gives a homotopy equivalence $S_{\text{walls}}(k+1,N) \tilde{\cong} BS(k,N)$. 
The hard part now is showing that $S(k+1,N)_o \sim \text{forget walls } S_{\text{walls}}(k+1,N)$ is a homotopy equivalence.

Previously this map was a surjective fibration with contractible fibers (convex combinations of all the walls that can be legally inserted).

Here the fibers are still contractible, but the map is no longer surjective: is not disjoint from any wall.

The image of $S_{\text{walls}}(k+1,N)$ is $S_{\text{disjoint}} = S_{\text{disjoint}}(k+1,N)$, consisting of surfaces in $S(k+1,N)$ disjoint from at least 1 wall.

We have:

$S(k+1,N)_o \leftarrow S_{\text{disjoint}}(k+1,N) \leftarrow S_{\text{walls}}(k+1,N)$ and need to show that $S_{\text{disjoint}}(k+1,N) \sim S(k+1,N)_o$.

For any surface $S \in S(k+1,N)_o$,
we will find a wall $W$ so that $h^W = 1_d$ and a homotopy $h^W$ and $h^W(S)$ is disjoint from $W$, giving a path from $S$ to $S_{\text{disjoint}}$.

But our construction will additionally satisfy the properties that:
- the same homotopy $h^W$ works for any surface $S'$ near to $S$
- $h^W$ is the identity outside a narrow slab around $W$

So for any $D^n \rightarrow S(k+1,N)_o$ with $D^n \subset S_{\text{disjoint}}$, by compactness of $D^n$ we can choose finitely many walls $W_i$ s.t. for each $S \in D^n$, at least one $h^{W_i}$ works.

Since the $h^{W_i}$ are disjointly supported, we can piece them together using a partition of unity on $D^n$, and get a homotopy $h$ of $D^n \rightarrow S(k+1,N)_o$ s.t. $h(D^n) \subset S_{\text{disjoint}}$.

This shows the relative homotopy group of $(S(k+1,N)_o, S_{\text{disjoint}})$ are trivial, so $S_{\text{disjoint}} \sim S(k+1,N)_o$ as desired.
How can we find such a wall \( W \) and homotopy \( h^w \)?

If we can find a \( (0,1)^{N-k-1} \) which is disjoint from \( S \),

We can push away from it in the \( R^k \) direction:

Whenever \( k+1 > d \), Sard’s theorem tells us that almost every \( (0,1)^{N-k-1} \) is disjoint from \( S \). So we only need to consider \( k+1 \leq d \), which for \( d=2 \) means only \( S(2,N) \):

1) First, expand in \( R^k \) direction as before:

2) Now unzip the middle of the product region near this slice (while keeping the rest of the surface fixed):

To do this, we need the intersection \( (0,1)^{N-k-1} \cap S \).

But recall that \( S \mapsto (0,1)^{N-k-1} \cap S \) induces the isomorphism \( \pi_0(S(k+1,N)) \cong \Omega_0 S(k+1,N) \), so what we need is exactly that \( S \in S(k+1,N) \).