Overview of course

Homological stability is the phenomenon in low-dimensional topology:

\[ H_i(B_n) \approx H_i(B_{n+1}) \text{ for } n \gg i \]
\[ H_i(S_n) \approx H_i(S_{n+1}) \text{ for } n \gg i \]
\[ H_i(\text{Aut}(F_n)) \approx H_i(\text{Aut}(F_{n+1})) \text{ for } n \gg i \]
\[ H_i(M_{0g}) \approx H_i(M_{0g+1}) \text{ for } g \gg i \]

But what does the homology converge to?
What are these stable homology groups?

Scanning is a tool, used by Galatius but with precursors in earlier work, for answering this question. The key step is to recognize a natural geometric space whose homology we converge to.

Today we’ll outline how to use scanning to prove that
\[ H_i(B_n) \text{ converges to } H_i(\Omega^2 S^1) \] (the subscript means take only one component)

braid group \( B_n \), elements look like \( \begin{array}{c} \infty \end{array} \) \[ \begin{array}{c} \infty \\ \vdots \\ \infty \end{array} \]

but for our purposes, we only care that \( B_n = \pi_1(X_n) \)

\[ X_n = \text{space of } n \text{ distinct unordered points in the plane} \]
\[ = \{ (x_1, \ldots, x_n) \in \mathbb{C}^n \mid z_i \neq z_j \} / S_n \]

\( X_n \) is aspherical, so \( H_i(B_n; \mathbb{Z}) = H_i(X_n; \mathbb{Z}) \)
$H_n(B^n;\mathbb{Z})$ converges to $H_1(S^2;\mathbb{Z})$.

What is the relation?

$X_n$ is the space of $n$ distinct unordered points in the plane. Put a $+1$ eV charge at each point; what is the force experienced by an electron?

$$ F \sim \frac{1}{r^2} $$

goes to $0$ at infinity, pole at each point

Sending $p$ to the vector $F_0$ yields a map $R^2 \to R^2 \cup \{0\}$ invert through unit circle, get a map $R^2 \cup \{0\} \to R^2 \cup \{0\}$ sending $\infty$ to $0$

$X_n \to \text{Maps}(S^2, S^2) \leftarrow \Omega^2 S^2$

AKA

It's ridiculous that $X_n \cong \Omega^2 S^2$ — ($X_n$ is finite-dimensional, $\Omega^2 S^2$ is not)

* $X_n$ is aspherical with $\pi_i = B^n$

* $S^2 \text{ is not and has } \pi_1 = \mathbb{Z}$

* $X_n \neq X_{n+1}$, so they can't be $\cong$ to same space

but let's try to prove it anyway.

Recall that $B$ takes a topological group $G$ and gives its classifying space $BG$ (maps $X \to BG$ — at same as $S^2$

Claim: $B$ is inverse to $S^2$.

Proof: $\pi_i(S^2 X) = \pi_{i+1}(X)$ ($\pi_0(S^2 X) = \pi_1(X)$ is definition of $\pi_i$; other cases same idea)

$\pi_1(G) = \pi_1(BG)$ ($\pi_{i+1}(BG) = \text{maps } S^i \to BG = G\text{-bundles over } S^{i+1}$ determined by clutching function from equator $S^i \to G)$

So $G \cong BG$ by Whitehead's theorem.

So to prove that $X_n \cong \Omega^2 S^2$, it would suffice to prove that $BBX_n \cong S^2$. 

Same map $X_n \to \text{Maps}(S^2, S^2)$, new perspective

Look through a microscope w/ autozoom
(field of vision = $\frac{1}{10}$ minimum distance between points)

At each point, we see through the viewfinder either one point, or nothing.
(the former becomes the latter as it slides out of the viewfinder / $\Delta$ viewfinder = $S^2$

It turns out the important part is not zooming, so much as accepting that our vision is limited.

$Z := \text{space of finite subsets of } [\square], \quad Z \cong S^2$

topologized so points can slide out of frame continuously

new camera owner = take snapshots everywhere
(doesn't matter how much you zoom in or out)

$X_n$

Between $X_n$ and $Z$ we can interpolate

$Y := \text{space of finite subsets of } [\square], \quad Y \cong S^2$

topologized so points can slide out of frame at top and bottom

To talk about $BX_n$, we need a "multiplication": CONCATENATION:

$[\square] + [\square] = [\square] \quad X := \bigsqcup X_n = \text{space of finite subsets of } [\square]$
It turns out that $Y$ really is $B\Sigma$, and $Z$ really is $BY$! 4)

so $BBX \approx Z \approx S^2$, and we get $X \approx \Omega^2 S^2$ ?!

N.D. When we proved that $G \approx \Omega S^2 G$, it was only for topological groups. Our spaces are only topological monoids, and it's impossible that $X \approx BBX \approx \Omega Y$

(e.g. components of $X$ have different fundamental groups $\pi_n \neq \pi_n$)

Ex. IN is a perfectly nice monoid. What is $\pi_1$ of $BIN = Kn(1,1)$? It's not IN, because $\pi_1$ is a group. (It turns out $BIN = K(2,1) = S^1$.)

It turns out that $G \approx \Omega S^2 G$ holds for any topological monoid where $\pi_0$ is a group. (Note that this implies all components of $G$ are the same)

For other $G$, all we can say about $G$ and $\Omega S^2 G$ is:

**Group Completion Theorem:** $H_i(\Omega S^2 G) = H_i(G)[\pi_0]$

what does this mean?

$G$ acts on itself by multiplication, factors through action of $\pi_0$ on $H_i(G)$.

**force** this action to be invertible

So: $H_i(X) = H_i(\bigcup X_n) = \bigoplus H_i(X_n)$

$H_i(X_0) \oplus H_i(X_1) \oplus H_i(X_2) \oplus H_i(X_3) \oplus \cdots$ ← inverting action of $\pi_0$

is shifting the other way

$H_i(X_1) \oplus H_i(X_2) \oplus H_i(X_3) \oplus \cdots$

$\pi_0$ acts by shifting this over

pretend there is a space $X_{\infty}$ whose homology is the stable homology

$H_i(X_0) = \lim_{n \to \infty} H_i(X_n)$

$H_i(X_\infty)[\pi_0]:$

$\oplus H_i(X) \oplus H_i(X_0) \oplus H_i(X_0) \oplus H_i(X_0) \oplus \cdots = H_i(\bigcup X_\infty)$

$H_i(\bigcup X_\infty) = H_i(X)[\pi_0]$ 5)e+ outraged $H_i(\Omega BX) = H_i(\Omega Y) = H_i(\Omega S^2 BY) = H_i(\Omega S^2 Z) = H_i(\Omega^2 S^2)$

$\lim_{n \to \infty} H_i(B_n; \mathbb{Z}) = \lim_{n \to \infty} H_i(K_n; \mathbb{Z}) = H_i(\Omega^2 S^2)$