Math 151 Midterm 1 Solutions

1. Suppose that 28 percent of the population drink tea, 35 percent drink coffee, 10 percent drink both. Find:
   (i) percentage of people who do not drink either.
   (ii) percentage of people who drink tea but not coffee
   **Solution.** (i) Answer: 47 percent (ii) Answer: 18 percent

2. Show that if events A, B, C are independent, then so are A and B ∪ C
   **Solution.**

3. Two hunters shoot at a deer, which is hit by exactly one bullet. If the first hunter hits targets with probability 0.3 and the second with probability 0.6, what is the probability the second hunter killed the deer? The answer is not 2/3.
   **Solution.**

4. Two dice are thrown n times. Find the smallest n so that the probability of the sum of dice 7 appearing at least once is greater than 1/2.
   **Solution.** Problem is equivalent to the following: Find the smallest n so that the probability of the sum of dice 7 never appearing is smaller than 1/2. There are 6 choices (1,6),(2,5),(3,4),(4,3),(5,2),(6,1) out of 36. Hence we need to find the least n to satisfy:
   \[ (\frac{30}{36})^n = (\frac{5}{6})^n < \frac{1}{2} \]
   \[ (\frac{30}{36})^2 = (\frac{5}{6})^2 = \frac{25}{36}, (\frac{5}{6})^3 = \frac{125}{216}, (\frac{5}{6})^4 = \frac{625}{1296} < \frac{1}{2} \]
   Hence, the least n is 4.

5. English and American spellings are rigour and rigor, respectively. A man staying at a Parisian hotel writes this word, and a letter taken at random from his spelling is found to be a vowel. If 40 percent of the English speaking men at the hotel are English and 60 percent are Americans, what is the probability that the writer is an Englishman?
   **Solution.** Let E=”Writer is an Englishman”, A=”Writer is an American”, V=”A random letter is a vowel”
   \[ P(E|V) = \frac{P(EV)}{P(V)} = \frac{P(E)V(E) + P(A)V(A)}{P(V)} \]
   \[ = \frac{0.4(0.5)}{0.4(0.5) + 0.6(0.2)} = \frac{20}{44} = \frac{5}{11} \]

6. Let \( Q_n \) denote the probability that in n tosses of a fair coin no run of 3 consecutive heads appears. Show that:
   (a) \( Q_n = \frac{1}{2}Q_{n-1} + \frac{1}{3}Q_{n-2} + \frac{1}{8}Q_{n-3} \) for \( n \geq 3 \)
   and \( Q_0 = Q_1 = Q_2 = 1 \)
   (b) Find \( Q_6 \)
   **Solution.**
   (a) \( A_n = \)”In n tosses of fair coin no run of 3 consecutive heads appear” First, we are conditioning on first trial. If tail appears, then it is equivalent to having 3 consecutive heads in remaining \( n - 1 \) trials. Similarly, if H, T appear in first two trials respectively, then it is equivalent to having 3 consecutive heads in remaining \( n - 2 \) trials. Repeating same argument for third trial, we get:
\[ Q_n = P(A_n) = P(H)P(A_n|H) + P(T)P(A_n|T) = \frac{1}{2}P(A_n|H) + \frac{1}{2}P(A_n|T) = \]
\[ \frac{1}{2}P(A_{n-1}) + \frac{1}{2}[P(H)P(A_{n-1}|HH) + P(T)P(A_{n-1}|HT)] = \frac{1}{2}P(A_{n-1}) + \frac{1}{2}P(A_{n-2}) + \frac{1}{4}P(A_n|HH) = \]
\[ \frac{1}{2}P(A_{n-1}) + \frac{1}{4}P(A_{n-2}) + \frac{1}{4}[P(H)P(A_{n-2}|HHH) + P(T)P(A_{n-2}|HHT)] = \]
\[ \frac{1}{2}P(A_{n-1}) + \frac{1}{4}P(A_{n-2}) + \frac{1}{4}[0 + \frac{1}{2}P(A_{n-3})] = \frac{1}{2}Q_{n-1} + \frac{1}{4}Q_{n-2} + \frac{1}{8}Q_{n-3} \]

(b) Answer: \[ \frac{24}{16} = \frac{11}{16} \]

7. The probability of the closing of the \( i \)-th gate in the circuits shown in the figure is given by \( p_i, i = 1, 2, 3, 4, 5, 6, 7 \). If all gates function independently, what is the probability that a current flows between A and B?

**Solution.** \( E=\text{"Current flows through gates 1,2,3"} \), \( F=\text{"Current flows through gates 1,4"} \), \( G=\text{"Current flows through gates 5,6,7"} \) Notice that current flows between A and B, only if one of events \( E,F,G \) happens.

Using independence and inclusion-exclusion principle:
\[ P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG) = \]
\[ p_1p_2p_3 + p_1p_4 + p_5p_6p_7 - p_1p_2p_3p_4 - p_1p_2p_3p_5p_6p_7 - p_1p_4p_5p_6p_7 + p_1p_2p_3p_4p_5p_6p_7 \]