Enumerative Algebraic Geometry

via

Techniques of Symplectic Topology

and

Analysis of Local Obstructions
Enumerative Geometry

Subject Matter
determine the number of geometric objects that satisfy given geometric conditions

Example
the number of lines through 2 points in Euclidean space is 1

Typical Setting
Objects: (complex) curves/Riemann surfaces in algebraic manifolds (e.g. $\mathbb{P}^n$)
Conditions: genus/complex structure
homology class
singularities
pass through submanifolds (e.g. pts)
Classical Example

Formulation

\( n_d = \# \) of rat. deg.-d curves thr. \( 3d - 1 \) pts. in \( \mathbb{P}^2 \)

What is \( n_d \)?

- rational genus=zero \( (S^2) \)
- degree-\( d \) \( [C] = d[\ell] \in H_2(\mathbb{P}^2; \mathbb{Z}) \)

Classical Results (by 1870s)

\( n_1 = 1, \quad n_2 = 1, \quad n_3 = 12, \quad n_4 = 620 \)

Recent Results (1993)
(Kontsevich-Manin, Ruan-Tian)

\[ n_d = \frac{1}{6(d-1)} \sum_{d_1 + d_2 = d} \left( d_1 d_2 - \frac{2(d_1 - d_2)^2}{3d - 2} \right) \left( \frac{3d - 2}{3d_1 - 1} \right) d_1 d_2 n_{d_1} n_{d_2} \]

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<td>( n_d )</td>
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<td>87,304</td>
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<td>26,312,976</td>
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also recursion for \( n_d(\mu) \), \( \mu = \) submanifolds in \( \mathbb{P}^n \)
Symplectic Topology

General Question
When are two symplectic manifolds equivalent?

**Pseudoholomorphic Curves** (Gromov’85)

\[(V, \omega)\text{=symplectic manifold, } A \in H_2(V; \mathbb{Z})\]

\[J\text{=compatible (almost) complex structure}\]

\[\mathcal{M}_{0,0}(V, A) = \left\{ u \in C^\infty(S^2, V) : u_*[S^2] = A, \right.\]
\[\left. \bar{\partial}_J u = 0 \right\} / PSL_2\]

**Symplectic Invariants**

*Fact:* \( \exists \overline{\mathcal{M}}_{0,0}(V, A) \) in good cases

*Corollary:* if \( \mu_1, \ldots, \mu_N \) \( = \) submanifolds in \( V \),

\[\# \left\{ [u] \in \overline{\mathcal{M}}_{0,0}(V, A) : \text{Im} \ u \cap \mu_l \neq \emptyset, \ l = 1, \ldots, N \right\}\]

depends only on \( \omega, A, \) and \( [\mu_l] \in H_\ast(V; \mathbb{Q}) \)
Example

\((\mathbb{P}^2, \omega, J) = \text{Fubini-Study structure}\)

\(\mathcal{M}_{0,0}(\mathbb{P}^2, d) = \{ u \in C^\infty(S^2, \mathbb{P}^2) : u_*[S^2] = d[\ell], \quad \bar{\partial}_J u = 0 \}/PSL_2 \)

\(p_1, \ldots, p_{3d-1} = \text{points in } \mathbb{P}^2\)

\(\# \{ [u] \in \overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, d) : p_l \in \text{Im } u \} = n_d \)

More Generally

\(\mathcal{M}_{0,N}(\mathbb{P}^n, d) = \{ (u; y_1, \ldots, y_N) : y_l \in S^2 \}/PSL_2 \)

Fact: \(\exists \) “nice” \(\overline{\mathcal{M}}_{0,N}(\mathbb{P}^n, d)\)

AG: Kontsevich’93, Fulton-Pandharipande’97

SG: McDuff-Salamon’93, Ruan-Tian’93

Fact (Pandharipande’95)

intersections of tautological classes in \(H^* (\overline{\mathcal{M}}_{0,N}(\mathbb{P}^n, d); \mathbb{Q})\) are computable

\(\{ \text{tautological classes} \} \supset \{ \text{relevant classes} \}\)
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Subject Matter

determine \# of geometric objects
that satisfy given geometric conditions

Typical Setting

Objects: (complex) curves/Riemann surfaces
in algebraic manifolds (e.g. $\mathbb{P}^n$)

Conditions: genus/complex structure
homology class
singularities
pass thr. submanifolds (e.g. pts)
Two Types of Problems

Problem 1
Determine \# of rational curves with the given uni-pointed singularities (e.g. cusp of specified form)
Goal: answer in terms of ITC

Problem 2
Determine \( n_{g,d}(\mu) = \# \) of genus-\( g \) curves with the given complex structure
Goal: answer in terms of ITC and genus-\( g \) symplectic invariants
Problem 1

Example

$$|S_1(\mu)| = \# \text{ deg.-d rat. curves with a cusp \ thr. } 3d-2 \text{ pts in } \mathbb{P}^2$$

What is $|S_1(\mu)|$?
Contribution to the Euler Class

Setup

\[ X \text{ cmpt} \\
\{V^n \leftarrow s \rightarrow X^{2n}\} \\
s \in \Gamma(X; V) \\
Z \subset X \]

What is \( C_Z(s) \)?

If \( s \pitchfork 0 \), \[ C_Z(s) = \pm |s^{-1}(0) \cap Z| \]

More Generally

\[ C_Z(s) \equiv \pm \left| \{s + \varepsilon\}^{-1}(0) \cap W_Z \right| \text{ if} \]
\[ \varepsilon \in \Gamma(X; V) \text{ small & generic} \]
\[ W_Z = \text{small neighborhood of } Z \]

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<td>( C_{Z_1}(s) = 3 )</td>
<td>( C_{Z_2}(s) = 1 )</td>
<td>( C_{Z_3}(s) = 2 )</td>
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\[ |S| = \langle e(V), [X] \rangle - \sum C_{Z_i}(s) \]
Computation of $C_Z(s)$

Setup
$V^n \rightarrow X^{2n}$, $X$ cmpt, $s \in \Gamma(X; V)$, $Z \subset X$

Definition of $C_Z(s)$
$$C_Z(s) \equiv \pm |\{s + \varepsilon\}^{-1}(0) \cap W_Z|$$
if
$\varepsilon \in \Gamma(X; V)$ small & generic
$W_Z =$ small neighborhood of $Z$

Easy
$C_Z(s)$ is well-defined if
$s^{-1}(0) \cap Z \& s^{-1}(0) - Z$ are closed

Proposition
$C_Z(s)$ is well-defined if
(i) $Z$ is smooth & $W_Z$ is modellable on $F \rightarrow Z$
(ii) $s|W_Z \approx$ (polynomial $\alpha: F \rightarrow V$)

$$C_Z(s) = \pm |\{v \in F: \bar{\nu}_v + \alpha(v) = 0\}|$$

for generic $\bar{\nu} \in \Gamma(Z; V)$

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Zeros of Polynomial Maps

Setup

\[ F^k \xrightarrow{\psi_{\alpha, \bar{\nu}}} \mathcal{O}^{k+m} \]

$X$ cmpt

$F = \bigoplus F_i, \quad \alpha = \sum \alpha_i$

$\alpha_i \in \Gamma(X; \text{Hom}(F_i \otimes d_i, \mathcal{O}))$

$\bar{\nu} \in \Gamma(X; F)$ generic w.r.t. $\alpha$

Facts

(1) $\pm |\psi_{\alpha, \bar{\nu}}^{-1}(0)|$ depends on $\alpha$, but not $\bar{\nu}$

(2) if $\alpha | F_x$ is injective $\forall x \in X$,

\[ \pm |\psi_{\alpha, \bar{\nu}}^{-1}(0)| = \langle e(\mathcal{O}/\alpha(F)), [X] \rangle = \langle c(\mathcal{O})c(F)^{-1}, [X] \rangle \]

Example

$X = \mathbb{P}^1 = \{ \ell = [u, v] : (u, v) \in \mathbb{C}^2 - \{0\} \}$

$F = \mathbb{C}, \quad \mathcal{O} = \mathbb{C} \oplus \gamma^*$

(1) if $\alpha(\ell; c) = (\ell; c, 0), \quad \pm |\psi_{\alpha, \bar{\nu}}^{-1}(0)| = 1$

(2) if $\alpha(\ell; 1) = (\ell; 0, c \cdot u), \quad \pm |\psi_{\alpha, \bar{\nu}}^{-1}(0)| = 0$
Computation of $\pm |\psi_{\alpha,\bar{\nu}}^{-1}(0)|$

\[(X, F, \mathcal{O}, \alpha)\]

\[X \rightarrow \mathbb{P}F_i \quad F_i \rightarrow \gamma^{d_i}\]

\[(X, F, \mathcal{O}, \alpha), \alpha \text{ is linear}\]

\[X \rightarrow \mathbb{P}F \quad F \rightarrow \gamma\]

\[(X, F, \mathcal{O}, \alpha), \alpha \text{ is linear} \& \text{rk } F = 1\]

$\pm |\psi_{\alpha,\bar{\nu}}^{-1}(0)| = \langle c(\mathcal{O})c(F)^{-1}, [X] \rangle - C_{\alpha^{-1}(0)}(\alpha^\perp)$,

$\alpha^\perp \in \Gamma(X; \text{Hom}(F, \mathcal{O}/\mathbb{C}\bar{\nu}))$

\[(X_j, F_j, \mathcal{O}_j, \alpha_j), \text{rk } \mathcal{O}_j < \text{rk } \mathcal{O}\]
Example 1

\[ |S_1(\mu)| = \# \text{ deg.-}d \text{ rat. curves with a cusp thr. } 3d-2 \text{ pts in } \mathbb{P}^2 \]

\[ |S_1(\mu)| = \langle 3a^2 + 3ac_1(L^*) + c_1^2(L^*), [\tilde{V}_1(\mu)] \rangle - |V_2(\mu)| \]

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<td>S_1(\mu)</td>
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<td>0</td>
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<td>12</td>
<td>2,304</td>
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Example 2

\[ |S_2(\mu)| = \# \text{ deg.-}d \text{ rat. curves with a (3, 4)-cusp thr. } 3d-4 \text{ pts in } \mathbb{P}^2 \]

\[ |S_2(\mu)| = \langle 33a^2c_1^2(L^*) + 18ac_1^3(L^*) + 4c_1^4(L^*), [\tilde{V}_1(\mu)] \rangle \]

\[ - \langle 21a^2 + 9a(c_1(L_1^*) + c_1(L_2^*)) \]

\[ + 2(c_1^2(L_1^*) + c_1^2(L_2^*)) + c_1(L_1^*)c_1(L_2^*), [\tilde{V}_2(\mu)] \rangle \]

\[ + 3|V_3(\mu)| \]

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<td>0</td>
<td>147</td>
<td>54,612</td>
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Extent of Applications

(1) count rat. curves w. specified cusp in \( \mathbb{P}^n \)

(2) should apply to \( G/P \) (e.g. \( Gr_k \mathbb{C}^n \))
   to get ITC’s
Two Types of Problems

Problem 1
Determine \# of rational curves with the given uni-pointed singularities (e.g. cusp of specified form)

Goal: answer in terms of ITC

Problem 2
Determine \( n_{g,d}(\mu) = \# \) of genus-\( g \) curves with the given complex structure

Goal: answer in terms of ITC and genus-\( g \) symplectic invariants
Problem 2

Genus One, \( \mathbb{P}^n \) (Ionel’96)

If \( \mu_1, \ldots, \mu_N \) are submanifolds in \( \mathbb{P}^n \),

\[
CR_1(\mu) \equiv RT_{1,d}(\mu_1; \mu_2, \ldots, \mu_N) - 2n_{1,d}(\mu),
\]

(1) is expressible in terms of \( \{n_{d'}(\mu')\} \)

(2) is \# of zeros of an affine map between vector bundles over \( \bar{V}_1(\mu) \)

\( RT_{g,d}(\cdot; \cdot) \) = sympl. genus-\( g \) invariant of \( (\mathbb{P}^n, \omega_{FS}) \)

as defined in Ruan-Tian’95
Genus $g \geq 2$

If $g = 2$ & $n = 2, 3$ or $g = 3$ & $n = 2$,

$$CR_g(\mu) \equiv RT_{g,d}(\mu_1; \mu_2, \ldots, \mu_N) - m(g) \cdot n_{g,d}(\mu),$$

(1) is # of zeros of affine maps between vector bundles over $\Sigma^j \times \tilde{S}_j(\mu)$, with $\tilde{S}_j(\mu) \subset \tilde{V}_{k_j}(\mu)$

(2) is expressible in terms of ITCs.

$$\boxed{g = 2, \ n = 2}$$

$$n_{2,d} = 3(d^2 - 1)n_d$$

$$+ \frac{1}{2} \sum_{d_1 + d_2 = d} \left( d_1^2d_2^2 + 28 - 16 \frac{9d_1d_2 - 1}{3d - 2} \right) \left( \frac{3d - 2}{3d_1 - 1} \right) d_1d_2n_{d_1}n_{d_2}$$

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<td>6,350,400</td>
<td>3,931,128,000</td>
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Symplectic & Enumerative Invariants

\[ \mathcal{H}_d(\mu) = \{ (y_1, \ldots, y_N; u) | u: \Sigma \rightarrow \mathbb{P}^n, u(y_l) \in \mu_l, \]
\[ u_*[\Sigma] = [u(\Sigma)] = d\ell, \]
\[ \bar{\partial} J,j u|_z = 0 \} \]

\[ m(g) \cdot n_{g,d}(\mu) = |\mathcal{H}_d(\mu)| \]

If \( \nu \in \Gamma(\Sigma \times \mathbb{P}^n; \pi^* T\Sigma \otimes \pi^*_{\mathbb{P}^n} T\mathbb{P}^n) \), let

\[ \mathcal{M}_{\nu,d}(\mu) = \{ (y_1, \ldots, y_N; u) | u: \Sigma \rightarrow \mathbb{P}^n, u(y_l) \in \mu_l, \]
\[ u_*[\Sigma] = d\ell, \]
\[ \bar{\partial} J,j u|_z = \nu|_{(z, u(z))} \} \]

For a generic \( \nu \), \( RT_{g,d} (; \mu) \equiv \pm |\mathcal{M}_{\nu,d}(\mu)| \)
Symplectic vs. Enumerative Invariants

If \( \nu_i \to 0 \) & \( (y_i, u_i) \in \mathcal{M}_{\nu_i, d}(\mu), \lim_{i \to \infty} (y_i, u_i) = \)

(1) \( b \in \mathcal{H}_d(\mu), \ OR \)

(2) \( b = (\Sigma_b, y, u), \Sigma_b = \Sigma \cup \bigcup S^2_h, u_b : \Sigma_b \to \mathbb{P}^n, \)

\( y_l \in \Sigma_b, u_b(y_l) \in \mu_l, \bar{\partial}u_b = 0 \)

\[ \Sigma_b = \]

\[ \begin{array}{c}
S^2 \\
\Sigma \\
S^2 \\
S^2 \\
\end{array} \]

\[ u_b \to \mathbb{P}^n \]
Symplectic vs. Enumerative Invariants

\[
\begin{align*}
\mathcal{H}_d &= \bar{\partial}^{-1}(0) \cap C^\infty \\
\mathcal{M}_{\nu,d} &= \{\bar{\partial} - \nu\}^{-1}(0)
\end{align*}
\]

\(\nu \in \Gamma(\bar{C}^\infty; F)\) small & generic

\[
\begin{array}{c}
\text{Computation of } C_{Z_i}(\bar{\partial}) \\
\text{Goal: reduce to counting zeros of a polynomial map} \\
\text{between finite-rank vector bundles over } \bar{Z}_i \\
\text{Method: obstruction-bundle approach (Taubes’84)} \\
\text{Norms: as in Li-Tian’96}
\end{array}
\]
Contribution to the Euler Class

Setup

\[ V^n \quad X \text{ cmpt} \]
\[ s \quad s \in \Gamma(X; V) \]
\[ X^{2n} \quad Z \subset X \]

What is \( C_Z(s) \)?

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If \( \nu_i \rightarrow 0 \& (y_i, u_i) \in \mathcal{M}_{\nu_i, d}(\mu) \), \( \lim_{i \rightarrow \infty} (y_i, u_i) = \)

(1) \( b \in \mathcal{H}_d(\mu) \), OR

(2) \( b = (\Sigma_b, y, u) \), \( \Sigma_b = \Sigma \cup \bigcup S^2_h \), \( u_b : \Sigma_b \rightarrow \mathbb{P}^n \), \( \bar{\partial} u_b = 0 \), \( y_l \in \Sigma_b \), \( u_b(y_l) \in \mu_l \), AND

(2a) \( u_b|\Sigma \) is simple & \( \Sigma_b \supset S^2 \), or

(2b) \( u_b|\Sigma \) is multiply-covered, or

(2c) \( u_b|\Sigma \) is constant.

\[ 23 \]
Potential Strata $Z_i$ in the $n=2$ Case

\[ d_1, d_2 > 0, \quad d_1 + d_2 = d \]
Extent of Applications

(1) $g \leq 7$ for $n = 2$; $g = 3$ for $n = 3$; $g = 2$ for $n = 4$

(2) might apply to $G/P$ to get ITC’s