In a single-variable calculus course, one learns how to find the derivative $y'$ of a smooth function $y=y(t)$. One also learns how to find an antiderivative $y=y(t)$ of a function $f=f(t)$. For example,

$$y'(t) = t^2 - \sin t,$$

$$\Rightarrow y(t) = \int (t^2 - \sin t)dt = \frac{1}{3}t^3 + \cos t + C, \quad t \in (-\infty, \infty),$$

for some real constant $C$.

Relation (1) is a differential equation. More generally, a differential equation is an equation satisfied by the derivatives of a function. Ordinary differential equations (ODEs), such as (1), involve functions of a single independent variable, which in the case of (1) is $t$. In contrast, partial differential equation (PDEs), such as

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial s^2}, \quad \text{where} \quad y=y(s,t),$$

involve functions of two or more variables. Differential equations, both ordinary and partial, arise naturally in mathematics, in physical and social sciences, and in engineering.

This course describes a variety of approaches to studying the behavior of solutions of ODEs. The primary focus will be on finding explicit solutions to ODEs, whenever possible. For example, (2) is the general solution to (1). The differential equation (1) has a particularly simple form:

$$y'(t) = f(t).$$

Any equation of this form can of course be solved by integration.

Most ODEs that describe real-life phenomena are far more complicated than (3). In this course, we will study a number of techniques for solving many ODEs, often by essentially reducing them to (3). These techniques build up on the chain and product rules of one-variable differentiation and the change-of-variables and integration-by-parts formulas of one-variable integration.

We start by considering first-order ODEs, i.e. ODEs that involve only the first derivative of a function. Linear first-order ODEs, i.e. equations of the form

$$y'(t) = a(t) \cdot y(t) + b(t),$$

can be reduced to (3) by multiplying both sides of (4) by an appropriate function $\mu=\mu(t)$, called an integrating factor. Thus, it is possible to write down an explicit solution $y=y(t)$ for (4).
Most first-order ODEs that arise in applications can be put into the normal form:

\[ y'(t) = Q(t, y(t)), \]  

(5)

where \( Q \) is a function of two variables. For example, in the case of (4),
\[ Q(t, y) = a(t) \cdot y + b(t). \]

If the function \( Q \) has a special form, it may be possible, using a technique from multivariable calculus, to find a function \( F \) of two variables such that
\[ F(t, y(t)) = 0, \]  

(6)

for all solutions of (5) and all admissible values of \( t \). While it may not be possible to solve (6) explicitly for \( y = y(t) \), (6) at least describes the graph of \( y \) in the \((t, y)\)-plane.

We will continue on to second-order ODEs, i.e. equations that involve the second derivative of a function such as
\[ y''(t) + p(t)y'(t) + q(t)y(t) = f(t). \]  

(7)

If the functions \( p(t) \), \( q(t) \), and \( f(t) \) are constant, the equation (7) can be solved explicitly for \( y = y(t) \). There are a number of ways of trying to approach (7) in general. One of them is the Laplace Transform, an operator that replaces (7) with a seemingly unrelated, but equivalent, equation, which may be far easier to solve.

We will see that most high-order ODEs can be replaced by equivalent systems of first-order ODEs, i.e. equations like (5), but for vector-valued functions \( y \). It is possible to find explicit solutions for a class of systems of ODEs. In other cases, some important qualitative information about solutions of a system can be obtained, even if it cannot be solved explicitly.

Many ODEs arising in applications can be solved numerically, for a fixed value of the parameter \( t \), quite accurately. Euler’s Method and Runge-Kutta Methods, that are motivated by the Taylor series expansion, lie behind many modern numerical ODE solvers. These methods will be studied in the course as well.