Math53: Ordinary Differential Equations  
Autumn 2004

Homework Assignment 6

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**Problem Set 6:**

9.2: 38*,40*,44; 9.4: 14; 9.6: 7,9; 9.7: 17; 9.8: 6,18,29; Problem E (see next page)

*Note:* In 9.2:38,40, sketch phase-plane portraits, as in Section 9.3.

**Daily Assignments:**

*Note:* In 9.2:38,40, sketch phase-plane portraits, as in Section 9.3.
Problem E

Recall that we are able to reduce the general first-order linear ODE

\[ y' + a(t)y = f(t), \quad y = y(t), \]

to a ready-to-integrate equation \((Py)' = Pf\) by finding an integrating factor \(P = P(t)\) such that

\[ P' = aP \quad \implies \quad (Py) = Py' + aPy. \]

Similarly, we can reduce a second-order linear ODE with constant coefficients

\[ y'' + py' + qy = f, \quad y = y(t), \quad p, q = \text{const}, \quad (1) \]

to a first-order linear ODE by multiplying by an integrating factor such that

\[ (P(y' + ay))' = P(y'' + py' + qy), \]

for some function \(a = a(t)\). This integrating factor is \(P(t) = e^{-\lambda_2 t}\), where \(\lambda_2\) is one of the roots of the corresponding characteristic polynomial \(\lambda^2 + p\lambda + q = 0\). We cannot adapt this approach to an arbitrary second-order linear ODE. Here is why.

(a) Suppose we would like to find smooth nonzero functions \(P = P(t)\) and \(Q = Q(t)\) such that

\[ (Q(y' + ay))' = P(y'' + py' + qy), \quad p = p(t), \quad q = q(t), \quad (2) \]

for some smooth function \(a = a(t)\) and for every smooth function \(y = y(t)\). Show that we must have

\[ P = Q, \quad P' + Pa = Pp, \quad \text{and} \quad (Pa)' = qP. \]

(b) Thus, the functions \(P\) and \(a\) can be found by finding a nonzero solution to

\[ \begin{pmatrix} P \\ (Pa) \end{pmatrix}' = \begin{pmatrix} p & -1 \\ q & 0 \end{pmatrix} \begin{pmatrix} P \\ (Pa) \end{pmatrix}, \quad P = P(t), \quad a = a(t). \]

Find a nonzero solution to this ODE if \(p\) and \(q\) are constant, obtaining an integrating factor for second-order ODEs with constant coefficients. Use it to find \(R_1 = R_1(t)\) and \(R_2 = R_2(t)\) such that

\[ (R_2(R_1y)')' = P(y'' + py' + qy), \quad p, q = \text{const}. \]

Express your final answer in terms of the roots \(\lambda_1\) and \(\lambda_2\) of the characteristic polynomial associated to the ODE (1).

(c) Apply the same approach to third-order ODEs. In other words, if \(p, q, r = \text{const}\), find functions \(P = P(t) \neq 0, \quad R_1 = R_1(t), \quad R_2 = R_2(t), \quad \text{and} \quad R_3 = R_3(t)\) such that

\[ (R_3(R_2(R_1y)')')' = P(y''' + py'' + qy' + ry). \]

Express your final answer in terms of the roots \(\lambda_1, \lambda_2, \text{and} \lambda_3\) of

\[ \lambda^3 + p\lambda^2 + q\lambda + r = 0. \]