Math53: Ordinary Differential Equations
Autumn 2004

Midterm II Solutions

Problem 1 (25pts)
Throughout this problem, \( y = y(t) \) denotes the solution to the initial-value problem
\[
y'' + y' - 2y = e^t, \quad y(0) = 1, \quad y'(0) = 2,
\]
and \( Y = Y(s) \) is the Laplace Transform of \( y = y(t) \).

(a; 5pts) Show that \( L y' = sY - 1 \) and \( L y'' = s^2Y - s - 2 \).

Since \( \{L f'(s)\} = s\{L f(s)\} - f(0) \) by the second table,
\[
L y' = sL y - y(0) = sY - 1
\]
\[
L y'' = sL y' - y'(0) = s(sY - 1) - 2 = s^2Y - s - 2.
\]

(b; 4pts) Show that \( Y(s) = \frac{1}{(s-1)^2(s+2)} + \frac{s+3}{(s-1)(s+2)} \).

Taking the Laplace Transform of both sides of the ODE defining \( y \) and using (a) and the first table, we obtain
\[
(s^2Y - s - 2) + (sY - 1) - 2Y = \frac{1}{s-1} \implies (s^2 + s - 2)Y - (s+3) = \frac{1}{s-1}
\]
\[
\implies Y(s) = \frac{1}{(s-1)^2(s+2)} + \frac{s+3}{(s-1)(s+2)} \quad \text{since} \quad (s^2 + s - 2) = (s-1)(s+2).
\]

(c; 16pts) Find the solution \( y = y(t) \) to the initial value problem
\[
y'' + y' - 2y = e^t, \quad y(0) = 1, \quad y'(0) = 2.
\]

By part (b),
\[
Y = L y = \frac{1}{(s-1)^2(s+2)} + \frac{s+1}{(s-1)(s+2)} = \frac{1}{(s-1)^2(s+2)} + \frac{1}{s-1} + \frac{1}{(s-1)(s+2)}.
\]

Applying quick partial fractions to the last fraction and then twice to the first, we obtain
\[
\implies Y = L y = \frac{1}{3(s-1)^2} + \frac{11}{9} \frac{1}{s-1} - 2 \frac{1}{9(s+2)} \implies y(t) = \frac{4}{3}te^t + \frac{11}{9} e^t - \frac{2}{9}e^{-2t}.
\]
Problem 2 (15pts)

(a; 8pts) Find all values of the constant \(c\) such that the origin in the xy-plane is a spiral sink or source for the solutions of the linear system

\[
y' = \begin{pmatrix} c & -1 \\ 4 & -1 \end{pmatrix} y.
\]

Specify whether the origin is a spiral source or a spiral sink for the values of \(c\) you find.

The origin is a spiral sink or source if and only if the eigenvalues of the matrix are complex, with nonzero real part. The characteristic polynomial in this case is

\[
\lambda^2 - (\text{tr } A)\lambda + (\det A) = \lambda^2 - (c-1)\lambda + (4-c)
\]

\[
\implies \lambda_1, \lambda_2 = \frac{1}{2}((c-1) \pm \sqrt{(c-1)^2 - 4(4-c)}) = \frac{1}{2}((c-1) \pm \sqrt{c^2 + 2c - 15}).
\]

Thus, we need

\[
c^2 + 2c - 15 = (c+5)(c-3) < 0 \quad \text{and} \quad c-1 \neq 0 \quad \implies \quad c \in (-5, 1), (1, 3).
\]

The origin is a \(\boxed{\text{spiral sink if } c \in (-5, 1)}\) since the real part of the eigenvalues is negative, and is a \(\boxed{\text{spiral source if } c \in (1, 3)}\) since the real part of the eigenvalues is positive.

(b; 7pts) Sketch the phase-plane portraits, in the xy-plane, for the system of ODEs in (a) with the values of \(c\) you found. Clearly show all important qualitative information, including the direction of rotation. Each of your sketches should contain at least two solution curves. Explain your reasoning.

If \(c \in (-5, 1)\), the origin is a spiral sink and the solution curves spiral \textit{in toward} the origin. If \(c \in (1, 3)\), the origin is a spiral source and the solution curves spiral \textit{out from} the origin. In both cases, the direction of rotation is \textit{positive}, i.e. counterclockwise, since the matrix entry in the lower-left corner is \textit{positive} (4 in this case).
Problem 3 (30pts)

(a; 20pts) Find the general solution \((x, y) = (x(t), y(t))\) to the system of ODEs
\[
\begin{align*}
  x' &= 8x - y \\
  y' &= 7y - 2x
\end{align*}
\]

This system of ODEs can be written as
\[
y' = \begin{pmatrix} 8 & -1 \\ -2 & 7 \end{pmatrix} y, \quad y = y(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.
\]

The characteristic polynomial for this matrix is
\[
\lambda^2 - (\text{tr } A)\lambda + (\text{det } A) = \lambda^2 - 15\lambda + 54 = (\lambda - 6)(\lambda - 9).
\]

Thus, the eigenvalues are \(\lambda_1 = 6\) and \(\lambda_2 = 9\). We first find an eigenvector \(v_1\) for \(\lambda_1\):
\[
\begin{pmatrix} 8 - \lambda_1 & -1 \\ -2 & 7 - \lambda_1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} 2c_1 - c_2 = 0 \\ -2c_1 + c_2 = 0 \end{cases} \iff c_2 = 2c_1 \implies v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.
\]

We next find an eigenvector \(v_2\) for \(\lambda_2\):
\[
\begin{pmatrix} 8 - \lambda_2 & -1 \\ -2 & 7 - \lambda_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} -c_1 - c_2 = 0 \\ -2c_1 - 2c_2 = 0 \end{cases} \iff c_2 = -c_1 \implies v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]

Thus, the general solution to the system of ODEs is
\[
y(t) = C_1 e^{6t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{9t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ or } \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} C_1 e^{6t} + C_2 e^{9t} \\ 2C_1 e^{6t} - C_2 e^{9t} \end{pmatrix}.
\]

(b; 8pts) Sketch the phase-plane portrait, in the \(xy\)-plane, for the system of ODEs in (a).
Since \(\lambda_1 < \lambda_2\), the slopes of the solution curves corresponding to \(C_1, C_2 \neq 0\) approach 2 as \(t \to -\infty\) and \(-1\) as \(t \to \infty\), but the curves do not approach the line \(y = -x\).

(c; 2pts) Determine whether the origin is a stable, asymptotically stable, or an unstable equilibrium point. Explain why.
Since one of the eigenvalues is positive (in fact, both are), some solution curves move away from the origin (in fact, all do). Thus, the origin is an \(\text{unstable}\) equilibrium.
Problem 4 (30pts)

(a; 15pts) Show that

\[ A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}, \text{ then } e^{tA} = e^{3t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \]

In the given case, the quickest way to verify the claim is to show that the claimed expression for \( e^{tA} \) solves the initial value problem:

\[
H' = AH, \quad H(0) = I ; \quad H(0) = e^{30} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \quad \checkmark \\
H' = 3e^{3t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} + e^{3t} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e^{3t} \begin{pmatrix} 3 & 1+3t \\ 0 & 3 \end{pmatrix} = e^{3t} \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = AH \quad \checkmark
\]

Here is a more direct approach. Since \( A \) has only one eigenvalue, \( \lambda = 3 \), we write

\[ A = \lambda I + (A-\lambda I) = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = B + C. \]

Since \( tB \) is diagonal, so is \( e^{tB} \) and the diagonal entries of \( e^{tB} \) are the exponentials of the diagonal entries of \( tB \):

\[ tB = \begin{pmatrix} 3t & 0 \\ 0 & 3t \end{pmatrix} \implies e^{tB} = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{pmatrix}. \]

On the other hand, since the only eigenvalue of \( tC \) is zero, the power series for \( e^{tC} \) terminates after \( n = 2 \) steps:

\[ e^{tC} = I + tC + \frac{t^2}{2!} C^2 + \frac{t^3}{3!} C^3 + \ldots = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \ldots = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \]

Finally, since \( tB = 3tI \) commutes with every matrix,

\[ (tB)(tC) = (tC)(tB) \implies e^{tA} = e^{tB+tC} = e^{tB} e^{tC} = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = e^{3t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \]

Another way of finding \( e^{tA} \) is to use \( e^{tA} = Y(t)Y(0)^{-1} \), where \( Y = Y(t) \) is a fundamental matrix for the ODE \( y' = Ay \). In other words, \( Y(t) = (y_1(t), y_2(t)), \) where \( y_1(t) \) and \( y_2(t) \) are any two linearly independent solutions of the ODE. This approach works for every \( A \).

(b; 15pts) Find the solution \( y = y(t) \) to the initial value problem

\[ y' = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} y - \begin{pmatrix} 3t \\ 3 \end{pmatrix}, \quad y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

Since \( t_0 = 0 \),

\[
y(t) = e^{tA}(y(0) + \int_0^t e^{-sA} f(s) \, ds) = e^{tA} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \int_0^t e^{-3s} \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3s \\ 3 \end{pmatrix} \, ds \right\} \\
= e^{tA} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t e^{-3s} \begin{pmatrix} 0 \\ 3 \end{pmatrix} \, ds \right\} e^{3t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} e^{-3t} = \begin{pmatrix} t \\ 1 \end{pmatrix}
\]