My area of research is the application of methods of algebraic topology to the study of loop spaces and path spaces, as well as moduli spaces of Riemann surfaces. In particular, I study string topology, which is a young field that examines topological structures suggested by string theory.

The area of string topology began with a construction by Chas and Sullivan [3] of previously undiscovered algebraic structure on the homology of the free loop space of a closed oriented manifold $M$. This is the space $LM$ of all continuous maps from the circle to $M$. Among other results, Chas and Sullivan found that the homology of $LM$, with its grading suitably shifted, carries the structure of a graded commutative algebra, much like the homology of an oriented manifold does by virtue of Poincaré duality.

This operation, the string loop product, can be understood by considering the space $\text{Map}(P, M)$ of maps from a pair-of-pants surface $P$ to $M$, where $P$ is thought of as a cobordism from a disjoint union of two circles to a circle. The cobordism $P$ is a model for the basic interaction in string theory, in which two closed strings merge into one string. There is a diagram

$$LM \times LM \xrightarrow{i} \text{Map}(P, M) \xrightarrow{j} LM,$$

where the maps $i$, $j$ are the restrictions to the incoming and outgoing boundary of $P$. The product is then constructed as the composition homomorphism $H_*(LM) \otimes H_*(LM) \xrightarrow{i^!} H_*(\text{Map}(P, M)) \xrightarrow{j^\ast} H_*(LM)$, where $i^!$ is an umkehr map.

Because the spaces involved are infinite-dimensional, the existence of $i^!$ is not immediate. It can be obtained by observing that $P$ is homotopy equivalent to a figure-eight graph $\Gamma = S^1 \vee S^1$, and that, for homology purposes, we may hence replace (1) by the diagram

$$LM \times LM \xrightarrow{i} \text{Map}(\Gamma, M) \xrightarrow{j} LM,$$

in which the maps are finite codimension embeddings in an appropriate sense. Using essentially diagram (2), Chas and Sullivan originally constructed the operation via transversality theory of smooth chains in $LM$. Since then, there have been other constructions, notably including the homotopy-theoretic approach by Cohen and Jones [5] in which $i^!$ is defined using a Thom-Pontrjagin collapse.

It was later shown by Cohen and Godin [4] that every connected cobordism $\Sigma$ between closed 1-manifolds having nonempty outgoing boundary yields an operation $\mu_\Sigma : H_*(LM)^{\otimes p} \to H_*(LM)^{\otimes q}$ which is compatible with gluing of cobordisms. This is a form of topological quantum field theory (TQFT). The construction of the $\mu_\Sigma$ exploits the fact that any surface $\Sigma$ may be represented as a fat graph, which is a finite graph $\Gamma$ endowed with enough extra structure to determine a surface with boundary having $\Gamma$ as a deformation retract.

The appearance of fat graphs reflects a result, due in its various incarnations to Harer [9], Penner [16], Strebel [21], and Igusa [10], that moduli spaces of Riemann surfaces with boundary are homotopy equivalent to certain spaces of metric fat graphs.
1 Current research

My dissertation [17] centers around the idea of open-closed string topology. In this context, an open string is a path in $M$ which need not be a loop. The basic spaces of open strings considered are of the form $PM(L_1, L_2)$, which stands for the space of paths $\gamma : [0, 1] \to M$ such that $\gamma(0) \in L_1$ and $\gamma(1) \in L_2$, where the $L_i$ are closed submanifolds of $M$. The $L_i$ are the so-called $D$-branes, or branes for short. In string theory, branes act as boundary conditions for open strings, which may interact among each other and with closed strings. It was Sullivan [22] who first sketched open-closed string topology operations.

A summary of my contributions is as follows. Fix throughout a manifold $M^n$ with a family of submanifolds $\{L_i\}_{i \in B}$ indexed by a set $B$.

- **Topology of fat $B$-graphs.** With the purpose of modeling the abstract interactions involving open and closed strings, I introduce the notion of fat $B$-graph.

The analogue of a cobordism in the presence of open strings is an open-closed cobordism, which is a cobordism between 1-manifolds with boundary equipped with a labeling of their endpoints by elements of $B$. More precisely, an open-closed cobordism is an oriented surface $\Sigma$ whose boundary has a decomposition $\partial \Sigma = \partial^- \Sigma \cup \partial^+ \Sigma \cup \partial_f \Sigma$ into incoming, outgoing and free boundary, which are embedded 1-manifolds with boundary. $\partial_f \Sigma$ is required to be a cobordism from $\partial^- \Sigma$ to $\partial^+ \Sigma$, and $\Sigma$ carries a labeling of the components of $\partial_f \Sigma$ by elements of $B$.

See Figure 1(a),(b) for examples. The $B$-cobordism in (a) is topologically a disk, with its boundary divided into two incoming intervals, an outgoing interval, and three free boundary intervals labeled by $1, 2, 3 \in B$. The one in (b) is topologically a surface of genus two and seven boundary components, which are decomposed as depicted.

![Figure 1: Two open-closed cobordisms and a fat $B$-graph, where $B = \{1, 2, 3, \ldots\}$.](image)

A fat graph is a finite graph with a specified cyclic ordering of the edges incident at each vertex. I defined a fat $B$-graph to be a fat graph in which each vertex is endowed with a finite set of brane labels drawn from $B$, which are placed in the cyclic ordering of edges. For an example, see Figure 1(c).

**Theorem 1** There is a space $F_B$ of metric fat $B$-graphs which is homotopy equivalent to $\coprod_{\Sigma} B\text{Diff}_B(\Sigma, [\partial])$, where $\Sigma$ ranges over isomorphism types of open-closed surfaces.
and \( \text{Diff}_B(\Sigma, [\partial]) \) is the group of structure-preserving diffeomorphisms of \( \Sigma \) which preserve each component of the boundary.

In Theorem 1 an open-closed surface is defined as an open-closed cobordism, minus the distinction between incoming and outgoing.

• **Open-closed string topology operations.** I show that fat \( B \)-graphs may be used to construct a rich class of open-closed string topology operations. More precisely, I prove the following.

**Theorem 2** Let \( \Sigma \) be an open-closed cobordism in which no connected component has empty outgoing boundary. Then, for any generalized homology theory \( h_* \) that supports orientations for \( M \) and the \( L_b \), there is a string topology operation

\[
\mu_\Sigma : h_*(\text{Map}_B(\partial^- \Sigma, M)) \to h_*(\text{Map}_B(\partial^+ \Sigma, M)).
\]

When \( \Sigma \) is an ordinary cobordism (so that the \( \partial^\pm \Sigma \) are circles), this operation agrees with the closed string topology operations in [4].

Here, each \( \text{Map}_B(\partial^\pm \Sigma, M) \) is the subspace of maps taking points labeled by \( b \in B \) into \( L_b \subseteq M \), and is homeomorphic to a cartesian product of copies of \( LM \) and spaces \( PM(L_a, L_b) \) of open strings. The construction is homotopy-theoretic, in the spirit of [5], and makes use of constrained mapping spaces from a fat \( B \)-graph to \( M \).

For example, if \( \Sigma \) is the \( B \)-cobordism from Figure 1(a), then \( \partial^- \Sigma = I_{1,2} \amalg I_{2,3} \) and \( \partial^+ \Sigma = I_{1,3} \), where \( I_{a,b} \) stands for an interval with endpoints labeled by \( a, b \in B \). Since \( \text{Map}_B(I_{a,b}, M) \cong PM(L_a, L_b) \), this yields an operation \( h_*(PM(L_1, L_2) \times PM(L_2, L_3)) \to h_*(PM(L_1, L_3)) \) (the string composition).

• **Open-closed topological quantum field theory.** To place Theorem 2 in a field-theoretic context, I define a notion of open-closed \( \text{TQFT} \), which is an analogue of two-dimensional \( \text{TQFT} \) in the presence of open strings.

Recall that a \( \text{TQFT} \) may be defined as a vector space \( V \) together with a homomorphism \( V^\otimes p \to V^\otimes q \) for each oriented cobordism from \( p \) circles to \( q \) circles. These maps satisfy certain coherence properties with respect to disjoint union and gluing of cobordisms.

An open-closed \( \text{TQFT} \) specifies a vector space \( V \) corresponding to the circle and a vector space \( V_{a,b} \) for every ordered pair \((a, b) \in B\), corresponding to an interval \( I_{a,b} \) with its endpoints labeled by \( a, b \in B \). For every open-closed cobordism \( \Sigma \) there is a homomorphism from the vector space corresponding to \( \partial^- \Sigma \) to the one corresponding to \( \partial^+ \Sigma \), and this assignment satisfies certain coherence properties. For example, the open-closed cobordism in Figure 1(a) gives rise to a map \( V_{1,2} \otimes V_{2,3} \to V_{1,3} \).

**Theorem 3** The string topology operations in Theorem 2 yield a positive boundary open-closed \( \text{TQFT} \) on the family \((h_*(LM), \{h_*(PM(L_a, L_b))\}_{a, b \in B})\).

Here, the qualifier positive boundary refers to the restriction that the connected open-closed cobordisms which yield operations must have nonempty outgoing boundary.
I formalize open-closed TQFT by extending the categorical notion of PROP (see for example [26]). With my generalization, an open-closed TQFT may be defined as a functor from a certain “\(B\)-PROP” of open-closed cobordisms into a \(B\)-PROP of vector space homomorphisms. The idea of field theory in the presence of open strings has been treated before by Segal [19], Moore [14], and Lazaroiu [11].

- **Explicit calculations.** Another aim is to develop tools for computing the string operations, and to compute them for interesting combinations of spaces and branes. The operation I analyze is the string composition operation, of the form \(H_*(PM(L_1, L_2)) \otimes H_*(PM(L_2, L_3)) \to H_*(PM(L_1, L_3))\). Using the Eilenberg-Moore spectral sequence, I make explicit calculations when \(M = \mathbb{C}P^n\) and the branes are projective spaces of lower dimension.

- **Higher-order string topology operations.** I show that the construction of open-closed string topology operations may be carried out “in families,” resulting in higher-order operations. More precisely, I show that there is a space \(\mathcal{CF}_B\) (consisting of metric fat \(B\)-graphs satisfying a technical condition) such that, letting \(\mathcal{CF}_\Gamma\) be the connected component of \(\Gamma \in \mathcal{CF}_B\), the following is true.

**Theorem 4** A class \(\alpha \in h_*(\mathcal{CF}_\Gamma)\), defines a string topology operation

\[
\Psi_{\Gamma, \alpha} : h_*(\text{Map}_B(\partial^- \Gamma, M)) \to h_*(\text{Map}_B(\partial^+ \Gamma, M))
\]

which agrees with the corresponding string topology operation \(\mu_{\Sigma}\) from Theorem 2 when \(\alpha \in h_0(\mathcal{CF}_\Gamma)\) is a generator.

For \(B = \emptyset\), it was conjectured in [4] that \(\mathcal{CF}_\Gamma\) is homotopy equivalent to a moduli space of Riemann surfaces. This would be an extension of the open-closed TQFT of Theorem 3 to an open-closed analogue of homological conformal field theory (HCFT) in the sense of Manin [13]. However, the conjecture has been disproved by Godin, so Theorem 4 yields only a partial HCFT. Refer to the section on future work for more on this connection.

- **The topology of the category of open and closed strings.** In joint work with Baas and Cohen [15], we studied the topology of a cobordism category \(\mathcal{S}^{oc}\) of open and closed strings. This is a 2-category in which the objects are compact one-manifolds whose boundary components are labeled by \(B\), the 1-morphisms are open-closed cobordisms, and the 2-morphisms are structure-preserving diffeomorphisms. Our methods and techniques are direct generalizations of those used by U. Tillmann ([24], [23]) in her study of the category of closed strings. We apply the striking theorem of Madsen and Weiss [12] regarding the topology of the stable mapping class group to identify the homotopy type of the geometric realization \(|\mathcal{S}^{oc}|\) as an infinite loop space. The result is:

**Theorem 5 ([15])** There is a homotopy equivalence

\[
\Omega |\mathcal{S}^{oc}| \simeq \Omega^\infty \mathbb{C}P_{-1}^\infty \times \prod_{b \in B} Q(\mathbb{C}P_+^\infty)
\]
of infinite loop spaces, where $X_+$ denotes $X$ with a disjoint basepoint, $\mathbb{C}P^\infty_-$ is the stunted projective spectrum (the Thom spectrum of minus the canonical bundle on $\mathbb{C}P^\infty$), and $Q(Y) := \lim_k \Omega^k \Sigma^k(Y)$.

- **Morse-theoretic description of string topology.** In the realm of closed string topology, I show that the string product may be described Morse-theoretically. This is accomplished by considering moduli spaces consisting of triples $(f_1, f_2, f_3)$ of smooth functions $f_1, f_2 : (-\infty, 0] \to LM$, $f_3 : [0, \infty) \to LM$ that satisfy gradient flow equations, and which are related at $t = 0$ by the property that the two loops $f_1(0), f_2(0) : S^1 \to M$ agree at $0 \in S^1$ and the loop $f_3(0)$ is their loop composition. The gradient flow equations are with respect to fixed choices of Morse-Palais-Smale functions on $LM$. By counting zero-dimensional moduli spaces of this type, one may define a product on the Morse complex of $LM$. I show in [17] that this product agrees with the Chas-Sullivan product.

This complements recent progress on the Floer cohomology $HF^*(T^*M)$ of the cotangent bundle with its canonical symplectic structure. A result due to Viterbo [25] (also proved using different methods by Salamon and Weber [18]), is that there is an additive isomorphism $HF^*(T^*M) \cong H_*(LM)$. Abbondandolo and Schwarz show in [1] that the product on $HF^*(T^*M)$ induced by counting $J$-holomorphic pairs of pants is carried by the isomorphism into the product sketched in the previous paragraph.

A corollary is then that there exists a ring isomorphism between $HF^*(T^*M)$ and the Chas-Sullivan ring $H^{*+n}_*(LM)$.

2 Future work

Open-closed string topology is an exciting new area, and there are many ideas that are yet to be applied in this setting. Some possible avenues of research are as follows.

- **A Hochschild description of open-closed string topology.** It is shown in [5] that there is a ring isomorphism $H^{*+n}_*(LM) \cong HH^*(C^*(M), C^*(M))$ between the Chas-Sullivan ring and the ring structure on the Hochschild cohomology of the cochain complex of $M$ induced by the cup product of cochains. It is therefore natural to ask if there is an analogous Hochschild description of the open string operations, such as the string composition $H_*(PM(L_1, L_2)) \otimes H_*(PM(L_2, L_3)) \to H_*(PM(L_1, L_3))$.

- **Deeper homotopy-theoretic structure.** Also in [5], it is shown that the string loop product has a very strong homotopy-theoretic realization, in the form of a ring spectrum structure on the Thom spectrum $LM^{-TM}$ arising from a certain operad action. Is there an analogue of this for the open string composition product?

- **$K$-theory computations.** Physical $D$-branes are often regarded as carrying extra information in the form of a $K$-theory class ([27],[8]). This makes it interesting to study open-closed string operations when the homology theory $h_*$ is $K$-theory.
• **Connections with Gromov-Witten theory.** The ring isomorphism of Floer homology $HF^*(T^*M)$ with the Chas-Sullivan ring of $LM$ suggests that there should be connections between string topology and Gromov-Witten invariants. In the closed case, Cohen and Eliashberg are pursuing a possible relationship between string topology and symplectic field theory. It would be interesting to find a connection between open-closed string topology and a form of relative Gromov-Witten invariants or relative symplectic field theory. In particular, one might expect a relation between open-closed string topology and the Legendrian contact homology of Ekholm, Etnyre and M. Sullivan [6]. Work in progress by Ekholm, Etnyre, Ng and M. Sullivan [7] may help provide a bridge between the two areas.

• **Homological conformal field theory.** As mentioned previously, the higher-order string operations $\mu_{\Gamma,\alpha}$ do not yield an hcft, because the spaces $\mathcal{CF}_{B,\Gamma}$ do not model the moduli spaces of Riemann surfaces. However, recent work by Bödigheimer [2] effectively builds a completion of this space which has a map into the moduli space. An interesting question is whether this completion makes it possible to construct a full hcft. Moreover, in Gromov-Witten theory and symplectic field theory, there is additional structure arising from the cohomology of the moduli space of curves. It would be reasonable to expect an hcft extension of string topology to relate to this structure.

• **Open-closed enriched elliptic objects.** The recent work by Stolz and Teichner [20] proposes a promising candidate for the objects that should play the role of chains in a geometric description of elliptic cohomology. It would be interesting to extend their ideas to a setting that incorporates open strings.

**References**


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Research Statement


