Homework #2 Solutions

Problems

- Section 2.3: 2, 10, 24, 32
- Section 2.4: 6, 18, 36 (the second domain should be $2 \leq x < 3$), 42
- Section 2.5: 4, 24, 36, 42 (do not submit graph)

2.3.2 The graphs of $f$ and $g$ are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.

(a) $\lim_{x \to 2} [f(x) + g(x)]$
(b) $\lim_{x \to 1} [f(x) + g(x)]$
(c) $\lim_{x \to 0} [f(x)g(x)]$
(d) $\lim_{x \to -1} \frac{f(x)}{g(x)}$
(e) $\lim_{x \to 2} [x^3 f(x)]$
(f) $\lim_{x \to 1} \sqrt{3 + f(x)}$

Solution:
(a) $\lim_{x \to 2} [f(x) + g(x)] = \lim_{x \to 2} f(x) + \lim_{x \to 2} g(x) = 2 + 0 = 2.$
(b) $\lim_{x \to 1^-} [f(x) + g(x)] = 2 + 2 = 4,$ but $\lim_{x \to 1^+} [f(x) + g(x)] = 2 + 1 = 3.$ Since the left- and right-hand limits exist but have different values, $\lim_{x \to 1} [f(x) + g(x)]$ does not exist.
(c) $\lim_{x \to 0} [f(x)g(x)] = (\lim_{x \to 0} f(x))(\lim_{x \to 0} g(x)) = 0(\lim_{x \to 0} g(x)) = 0$
(d) Since $\lim_{x \to -1} g(x) = 0$ but $\lim_{x \to -1} f(x) = -1 \neq 0,$ $\lim_{x \to -1} \frac{f(x)}{g(x)}$ does not exist.
(e) $\lim_{x \to 2} [x^3 f(x)] = (\lim_{x \to 2} x^3)(\lim_{x \to 2} f(x)) = (2^3)(2) = 16.$
(f) $\lim_{x \to 1} \sqrt{3 + f(x)} = \sqrt{3 + \lim_{x \to 1} f(x)} = \sqrt{3 + 1} = 2.$
2.3.10 Evaluate \( \lim_{x \to 4} \frac{x^2 - 4x}{x^2 - 3x - 4} \).

Solution: Substituting \( x = 4 \) directly produces 0/0, so we instead simplify:

\[
\lim_{x \to 4} \frac{x^2 - 4x}{x^2 - 3x - 4} = \lim_{x \to 4} \frac{x(x - 4)}{(x + 1)(x - 4)} = \lim_{x \to 4} \frac{x}{x + 1} = \frac{4}{5}
\]

2.3.24 Evaluate \( \lim_{x \to -4} \frac{\sqrt{x^2 + 9} - 5}{x + 4} \).

Solution: Substituting \( x = -4 \) directly produces 0/0, so we instead simplify using the conjugate radical:

\[
\lim_{x \to -4} \frac{\sqrt{x^2 + 9} - 5}{x + 4} = \lim_{x \to -4} \frac{\sqrt{x^2 + 9} - 5}{x + 4} \cdot \frac{\sqrt{x^2 + 9} + 5}{\sqrt{x^2 + 9} + 5} = \lim_{x \to -4} \frac{x^2 + 9 - 25}{(x + 4)(\sqrt{x^2 + 9} + 5)} = \lim_{x \to -4} \frac{x - 4}{(x + 4)(\sqrt{x^2 + 9} + 5)} = \lim_{x \to -4} \frac{x - 4}{x + 4} \cdot \frac{1}{\sqrt{x^2 + 9} + 5} = \frac{-8}{5 + 5} = -\frac{4}{5}.
\]

2.3.32 Prove that \( \lim_{x \to 0^+} \sqrt{x} e^{\sin(\pi/x)} = 0 \).

Solution: We use the Squeeze Theorem: first, note that for all \( x \),

\[-1 \leq \sin \left( \frac{\pi}{x} \right) \leq 1.\]

Exponentiating preserves inequalities, so

\[
\frac{1}{e} \leq e^{\sin(\pi/x)} \leq e.
\]

For \( x > 0 \), \( \sqrt{x} > 0 \), so multiplying through by \( \sqrt{x} \) preserves the inequalities:

\[
\frac{1}{e} \sqrt{x} \leq \sqrt{x} e^{\sin(\pi/x)} \leq e \sqrt{x}.
\]

Since \( \lim_{x \to 0^+} \frac{1}{e} \sqrt{x} = 0 \) and \( \lim_{x \to 0^+} e \sqrt{x} = 0 \), the Squeeze Theorem allows us to conclude that

\[
\lim_{x \to 0^+} \sqrt{x} e^{\sin(\pi/x)} = 0.
\]
2.4.6 Sketch the graph of a function \( f \) that is continuous except for the stated discontinuity: Discontinuities at \(-1\) and \(4\), but continuous from the left at \(-1\) and from the right at \(4\).

Solution: Answers may vary, but here is one example:

![Graph of function](image)

2.4.18 Explain why the function is discontinuous at the given number \( a \). Sketch the graph of the function.

\[
 f(x) = \begin{cases} 
 2x^2 - 5x - 3, & x \neq 3, \\
 6, & x = 3 
\end{cases}
 a = 3
\]

Solution: We first evaluate \( \lim_{x \to 3} f(x) \):

\[
 \lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{2x^2 - 5x - 3}{x - 3} = \lim_{x \to 3} \frac{(2x + 1)(x - 3)}{x - 3} = \lim_{x \to 3} (2x + 1) = 7
\]

Since the limit exists and has the value 7, but \( f(3) = 6 \), the function has a removable discontinuity at 3. Here is a graph:

![Graph of function](image)
2.4.36 Find the values of $a$ and $b$ that make $f$ continuous everywhere.

\[ f(x) = \begin{cases} 
  x^2 - 4, & x < 2, \\
  x - 2, & 2 \leq x < 3, \\
  ax^2 - bx + 3, & x \geq 3.
\end{cases} \]

Solution: We note that each of the functions is continuous on the domains specified, so we check that their left- and right-hand limits agree at the boundaries. First, at $x = 2$,

\[
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2^-} (x + 2) = 4 \quad \text{and} \quad \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} ax^2 - bx + 3 = 4a - 2b + 3,
\]

so for $f$ to be continuous at 2, $4a - 2b + 3 = 4$, and $4a - 2b = 1$. Next, at $x = 3$,

\[
\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} ax^2 - bx + 3 = 9a - 3b + 3 \quad \text{and} \quad \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 2x - a + b = 6 - a + b.
\]

Thus, for $f$ to be continuous at 3, $9a - 3b + 3 = 6 - a + b$, so $10a - 4b = 3$.

From the first equation, $b = 2a - 1/2$. Substituting this into the second equation gives $10a - 8a + 2 = 3$, so $2a = 1$, and $a = 1/2$. Backsolving for $b$ gives $b = 2(1/2) - 1/2 = 1/2$ as well.

2.4.42 Use the Intermediate Value Theorem to show that there is a root of $\sqrt[3]{x} = 1 - x$ in the interval $(0, 1)$.

Solution: We note that $f(x) = \sqrt[3]{x} + x - 1$ is continuous at all real numbers. Furthermore, $f(0) = 0 + 0 - 1 = -1$, and $f(1) = 1 + 1 - 1 = 1$. Since 0 is between 1 and $-1$, the Intermediate Value Theorem says that there is some number $c$ in $(0, 1)$ such that $f(c) = 0$. Then $\sqrt[3]{c} + c - 1 = 0$, so $\sqrt[3]{c} = 1 - c$. 

For the function \( g \) whose graph is given, state the following:

(a) \( \lim_{x \to \infty} g(x) \)
(b) \( \lim_{x \to -\infty} g(x) \)
(c) \( \lim_{x \to 3} g(x) \)
(d) \( \lim_{x \to 0^+} g(x) \)
(e) \( \lim_{x \to -2^+} g(x) \)
(f) The equations of the asymptotes

**Solution:**

(a) \( \lim_{x \to \infty} g(x) = 2 \).
(b) \( \lim_{x \to -\infty} g(x) = -2 \).
(c) \( \lim_{x \to 3} g(x) = +\infty \).
(d) \( \lim_{x \to 0^+} g(x) = -\infty \).
(e) \( \lim_{x \to -2^+} g(x) = -\infty \).
(f) The vertical asymptotes are \( x = -2, x = 0, \) and \( x = 3 \), and the horizontal asymptotes are \( y = 2 \) and \( y = -2 \).

**2.5.24** Find the limit \( \lim_{t \to -\infty} \frac{t^2 + 2}{t^3 + t^2 - 1} \).

**Solution:** Since both the numerator and denominator grow without bound as \( t \to -\infty \), we divide both by the largest power of \( t \) in the denominator:

\[
\lim_{t \to -\infty} \frac{t^2 + 2}{t^3 + t^2 - 1} = \lim_{t \to -\infty} \frac{\frac{t^2}{t^3} + \frac{2}{t^3}}{\frac{1}{t} + \frac{1}{t} - \frac{1}{t^3}} = \lim_{t \to -\infty} \frac{\frac{1}{t} + \frac{2}{t^3}}{\frac{1}{t} + \frac{1}{t} - \frac{1}{t^3}} = \frac{0 + 0}{1 + 0 + 0} = 0.
\]

**2.5.36** Find the limit \( \lim_{x \to (\pi/2)^+} e^{\tan x} \).

**Solution:** Let \( t = \tan x \). Then as \( x \to (\pi/2)^+ \), \( t \to -\infty \), so by the continuity of \( e^t \),

\[
\lim_{x \to (\pi/2)^+} e^{\tan x} = \lim_{t \to -\infty} e^t = 0.
\]
2.5.42 Find the horizontal and vertical asymptotes of the curve $y = \frac{2e^x}{e^x - 5}$.

Solution: We first find the horizontal asymptotes. As $x \to +\infty$,

$$
\lim_{x \to +\infty} \frac{2e^x}{e^x - 5} = \lim_{x \to +\infty} \frac{2e^x}{e^x} \cdot \frac{1}{e^{-x}} = \lim_{x \to +\infty} \frac{2}{1 - 5e^{-x}} = \frac{2}{1 - 5(0) - 1} = 2,
$$

so $y = 2$ is a horizontal asymptote. As $x \to -\infty$,

$$
\lim_{x \to -\infty} \frac{2e^x}{e^x - 5} = \frac{2(0)}{0 - 5} = 0,
$$

so $y = 0$ is also a horizontal asymptote.

We see that $\frac{2e^x}{e^x - 5}$ is continuous where it is defined. The only $x$-value where it is not defined is where $e^x - 5 = 0$, so $e^x = 5$, and $x = \ln 5$. As $x \to \ln 5$, the denominator goes to 0, but the numerator goes to $2e^{\ln 5} = 10 \neq 0$, so there is an infinite discontinuity at $x = \ln 5$. Thus, $x = \ln 5$ is the only vertical asymptote of this graph.