We could have alternatively defined direct sums and direct products through their universal properties, as follows:

A direct sum of two modules \(X_1, X_2\) is a module \(M\) together with homomorphisms \(f_i : X_i \to M\) (for \(i = 1, 2\)) such that, for any other module \(N\) and homomorphisms \(f'_i : X_i \to N\) there is a unique homomorphism \(F : M \to N\) so that \(f'_i = F \circ f_i\).

A direct product of two modules \(X_1, X_2\) is a module \(M\) together with homomorphisms \(f_i : M \to X_i\) (for \(i = 1, 2\)) such that, for any other module \(N\) and homomorphisms \(f'_i : N \to X_i\), there is a unique homomorphism \(F : N \to M\) such that \(f'_i = f'_i \circ F\).

Hand in one of problems 1, 2, 3, as well as all of problems 4–8.

(1) Let \(X_1, X_2\) be two \(R\)-modules. Write out the proof that \(X_1 \times X_2\), with coordinate-wise addition and scalar multiplication, together with the homomorphisms

\[
f_1 : x_1 \in X_1 \mapsto (x_1, 0) \in X_1 \times X_2\quad \text{and}\quad f_2 : x_2 \in X_2 \mapsto (0, x_2) \in X_1 \times X_2
\]

is a direct sum.

(2) Let \(X_1, X_2\) be two \(R\)-modules. Write out the proof that the set \(X_1 \times X_2\), with coordinate-wise addition and scalar multiplication, together with the homomorphisms

\[
f_1 : (x_1, x_2) \in X_1 \times X_2 \mapsto x_1 \in X_1\quad \text{and}\quad f_2 : (x_1, x_2) \in X_1 \times X_2 \mapsto x_2 \in X_2.
\]

is a direct product.

(3) Let \(X_1, X_2\) be two \(R\)-modules. Write out a proof that a direct sum is unique up to unique isomorphism.

(4) Take the universal property of “direct sum,” but replace modules by “groups”, and module homomorphisms by “group homomorphisms”.

Prove that in this context a direct sum of \(\mathbb{Z}/2\mathbb{Z}\) and \(\mathbb{Z}/2\mathbb{Z}\) is given by the infinite group of matrices \(\begin{pmatrix} \pm 1 & x \\ 0 & 1 \end{pmatrix}\) where \(f_1\) sends the nontrivial element of \(\mathbb{Z}/2\mathbb{Z}\) to \(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\) and \(f_2\) sends the nontrivial element of \(\mathbb{Z}/2\mathbb{Z}\) to \(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\).

(In particular, the result is neither free nor abelian.)

(5) Now replace modules by “sets” and homomorphisms by “functions.”

Prove that the direct sum of sets \(X_1, X_2\) is now given by the disjoint union \(X_1 \cup X_2\), where the functions \(f_1, f_2\) are given by the inclusions of \(X_1\) and \(X_2\).

For the next two problems: Recall that a tensor product \(M \otimes_R N\) of two \(R\)-modules \(M\) and \(N\) is equipped with a bilinear form \(M \times N \to M \otimes_R N\), which we denote by \((m, n) \mapsto m \otimes n\). An element of \(M \otimes_R N\) in the image of this map is called a pure tensor.

(6) Let \(A\) be a finite abelian group of order \(p\) and \(B\) a finite abelian group of order \(q\). Prove that \(A \otimes \mathbb{Z} B = 0\) if \(p\) and \(q\) are relatively prime. Prove also that \(\mathbb{Q} \otimes \mathbb{Z} (\mathbb{Q}/\mathbb{Z})\) is zero. (Tricky:) prove that \(\mathbb{R} \otimes \mathbb{Z} (\mathbb{R}/\mathbb{Z})\) is not zero.
(7) Let $M, N$ be $R$-modules, and let $K$ be a submodule of $N$.

(i) Show that there is a unique homomorphism $M \otimes K \to M \otimes N$ which sends $m \otimes k$ to $m \otimes k$ for every $m \in M, k \in K$.

(ii) Show this homomorphism need not be injective. (Consider the case of $R = \mathbb{Z}, M = \mathbb{Z}/2\mathbb{Z}, N = \mathbb{Z}$, and $K$ the even integers inside $N$).

(iii) Prove that the quotient of $M \otimes N$ by the image of $M \otimes K$ is isomorphic to $M \otimes (N/K)$. (Hint: call $Q$ this quotient. Construct a bilinear map $M \times N \to Q$ and show it has the correct universal property).

(8) Now take $M = \mathbb{C}^2, N = \mathbb{C}^3$ as $\mathbb{C}$-modules. Give an element of $\mathbb{C}^2 \otimes \mathbb{C}^3$ which isn’t a pure tensor. Prove, however, that any element of $\mathbb{C}^2 \otimes \mathbb{C}^3$ is the sum of two pure tensors.