Problem 1. Chapter 5 - ex 4: Suppose $S, T \in L(V)$ are such that $ST = TS$. Prove that $\ker(T - \lambda I)$ is invariant under $S$ for every $\lambda \in \mathbb{F}$.

Proof. Suppose $u \in \ker(T - \lambda I)$ ie $Tu = \lambda u$.
We want to show that $Su \in \ker(T - \lambda I)$ also ie $(T - \lambda I)Su = 0$.

$$TSu = STu = S(\lambda u) = \lambda Su \Rightarrow (T - \lambda I)Su = 0$$

Problem 2. Chapter 5 - ex 5.
Define $T \in L(F^2)$ by $T(w, z) = (z, w)$. Find all eigenvalues and eigenvectors of $T$.

Proof.

$$T(z, z) = (z, z); T(z, -z) = (-z, z) = -(z, -z)$$

$(1, 1), (1, -1)$ are two linearly independent eigenvectors with corresponding eigenvalues 1 and $-1$. The vector space is of dim 2. Hence there are 2 eigenvalues 1 and -1.

The eigenvectors associated with 1 are multiples of $(1, 1)$. The eigenvectors associated with $-1$ are multiples are $(1, -1)$.

Problem 3. Chapter 5 - ex 6: Define $T \in L(F^3)$ by :

$$T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$$

Find all eigenvalues and eigenvectors of $T$.

Proof. Denote by $M$ the matrix representation of $T$ with respect to the standard basis.

$$
\begin{pmatrix}
0 & 2 & 0 \\
0 & 0 & 0 \\
0 & 0 & 5
\end{pmatrix}
$$

The matrix is in upper triangular form, so we can read the eigenvalues off the diagonal, thus the eigenvalues are : 0 and 5.
The eigenvectors for eigenvalue 0 are in the null space of $T$, which is of dimension 1. Hence they are all multiples of $(1, 0, 0)$.

The eigenvectors for eigenvalue 5 are in the null space of $T - 5I$, whose matrix representation is, with respect to the standard basis:

$$
\begin{pmatrix}
-5 & 2 & 0 \\
0 & -5 & 0 \\
0 & 0 & 0
\end{pmatrix}
$$

Thus the null space in this case is of dimension 1. And all eigenvectors associated with eigenvalue 5 are multiples of $(0, 0, 1)$.

**Problem 4.** Chapter 5 - ex 7. Suppose $n$ is a positive integer and $T \in L(\mathbb{F}^n)$ is defined by

$$
T(x_1, \ldots, x_n) = (x_1 + \ldots + x_n, \ldots, x_1 + \ldots + x_n)
$$

Find all eigenvectors and eigenvalues for $T$.

**Proof.** Denote by $M$ the matrix representation of $T - \lambda I$ with respect to the standard basis:

$$
\begin{pmatrix}
1 - \lambda & 1 & 1 & \ldots & 1 \\
1 & 1 - \lambda & 1 & \ldots & 1 \\
1 & 1 & 1 - \lambda & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \ldots & 1 - \lambda
\end{pmatrix}
$$

in other words, $M$ is the matrix with $1 - \lambda$ along the diagonal and 1 elsewhere.

To compute the characteristic equation of $T$, we compute the determinant of $M$. First we carry out some row and column operations to put $M$ into upper triangular form.

1. For each $i \neq 1$, replace $i^{th}$ row with itself and $-1$ the first row. By this way, we obtain a matrix of the form: the first row remains the same; for each $i \neq 1$, $i^{th}$ row consists of $\lambda$ in the first entry, $-\lambda$ in the $i^{th}$ spot, and 0 otherwise.

$$
\begin{pmatrix}
1 - \lambda & 1 & 1 & \ldots & 1 \\
\lambda & -\lambda & 0 & \ldots & 0 \\
\lambda & 0 & -\lambda & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\lambda & 0 & 0 & \ldots & -\lambda
\end{pmatrix}
$$
2. Operating column operation on this new matrix by: replace with first column by itself with \(-1 \times (C_2 + \ldots + C_n)\), which gives us the matrix:

\[
\begin{pmatrix}
    n - \lambda & 1 & 1 & \ldots & 1 \\
    0 & -\lambda & 0 & \ldots & 0 \\
    0 & 0 & -\lambda & \ldots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \ldots & -\lambda
\end{pmatrix}
\]

Hence the characteristic polynomial for \(T\) is:

\[f(x) = (-1)^n(x - n)x^{n-1}\]

Thus the eigenvalues are \(n\) and 0.

The dimension of the null space for \(T - nI\) is 1, hence all eigenvectors associated with eigenvalue \(n\) are multiples of \((1, \ldots, 1)\).

The dimension of the null space for \(T\) is \(n - 1\) and the eigenvectors associated with 0 are of the form: \((-x_2 + \ldots + x_n), x_2, \ldots, x_n\), \(x_i \in \mathbb{F}\), ie the null space is spanned by:

\((-1,1,\ldots,0), (-1,0,1,\ldots,0), \ldots, (-1,0,\ldots,1)\)

\(\square\)

**Problem 5.** Chapter 5 - ex 11. \(V\) is a finite dim VS. Suppose \(S,T \in L(V)\). Prove that \(ST,TS\) have the same eigenvalues.

**Proof.**

1. Suppose \(\lambda\) is a nonzero eigenvalue for \(ST\), ie, \(\exists 0 \neq u \in V, STu = \lambda u\). From this equation, note that \(Tu \neq 0\). Hence apply \(T\) to both sides, we obtain:

\[TSTu = \lambda Tu \Rightarrow TS(Tu) = \lambda(Tu)\]

Since \(Tu \neq 0\), \(\lambda\) is thus an eigenvalue for \(TS\).

2. Suppose \(ST\) has 0 as an eigenvalue. WTS \(TS\) also has 0 as an eigenvalue, ie Ker\(TS\) is nontrivial.

Suppose Ker\(TS\) is trivial. Since \(V\) is finite-dim, \(TS\) is a linear isomorphism, ie \(TS\) is invertible, which by a previous hw problem, we have both \(T\) and \(S\) are invertible. This in turn gives that \(ST\) is invertible: this contradicts \(ST\) has a 0 as one of its eigenvalue, ie nontrivial kernel.
Hence we have shown that if $\lambda$ is an eigenvalue for $ST$, it is also one for $TS$. For the other direction, simply exchange the role of $T$ and $S$.

**Problem 6.** Chapter 5 – ex 12. Suppose $T \in L(V)$ is such that every vector in $V$ is an eigen vector of $T$. Prove that $T$ is a scalar multiple of the identity operator.

*Proof.* From the hypothesis, we have $\forall u, v \in V, \exists \lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}$ such that:

$$Tu = \lambda_1 u; Tv = \lambda_2 v$$

$$T(u + v) = \lambda_3(u + v)$$

Suppose that, contrary to hypothesis, $T$ is not a multiple of the identity. This means that we may choose $u, v$ so that $\lambda_1 \neq \lambda_2$. Then:

$$\lambda_3(u + v) = \lambda_1 u + \lambda_2 v$$

$$\Rightarrow (\lambda_3 - \lambda_1)u + (\lambda_3 - \lambda_2)v = 0$$

This means that any two vectors in $V$ are linearly dependent. Thus $V$ is one dimensional. In this case, every linear transformation from $V$ to $V$ (without any further conditions) is a multiple of the identity operator, contradiction.

**Problem 7.** Chapter 5 – ex 21. Suppose $P \in L(V)$ and $P^2 = P$. Prove that $V = \text{Ker } P \oplus \text{Range } P$.

*Proof.*

1. We have

$$v \in \text{Ker } P \cap \text{Range } P \Rightarrow \exists u \in V : v = Pu$$

$$\Rightarrow 0 = Pv = P^2 u = Pu = v \Rightarrow v = 0 \Rightarrow \text{Ker } P \cap \text{Range } P = \{0\}$$

2. For $v \in V$, we have $v = P(v - PV)$. Note that $P(v - PV) = PV - P^2 v = PV - PV = 0$. Hence $v - PV \in \text{Ker } P$

Thus $V = \text{Ker } P + \text{Range } P$

Combining two steps, we have the desired result.

**Problem 8.** Chapter – ex 23. Give an example of an operator $T \in L(\mathbb{R}^4)$ such that $T$ has no real eigenvalues.
Proof. Consider $T$ as operation defined by the multiplication on the left of column vectors by the following matrix.

\[
\begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

Then the characteristic polynomial for $T$ is the $\det T - xI$, is the product of det of two block matrices, each has $\det x^2 + 1$:

\[
\begin{pmatrix}
-x & -1 & 0 & 0 \\
1 & -x & 0 & 0 \\
0 & 0 & -x & -1 \\
0 & 0 & 1 & -x
\end{pmatrix}
\]

Hence characteristic polynomial $(x^2 + 1)^2$, which has no real roots. Since if $T$ has any eigenvalue, it has to be one of the roots of the characteristic polynomial. In this case, $T$ must have no (real) eigenvalues.

\[\square\]

**Problem 9.** Extra problem – Suppose that the characteristic polynomial $p(x)$ of a $3 \times 3$ matrix $M$ is $\det(xId - M) = (x - 1)(x - 2)(x - 3)$. What is the determinant of $M$? What is the characteristic polynomial of $M^{100}$?

**Proof.**

We know that $\text{Det } M = (-1)^3 p(0) = 6$.

From the characteristic polynomial for $M$, we know $M$ has 3 distinct eigenvalues 1, 2, 3, which comes with 3 corresponding linearly independent eigenvectors, called $v_1, v_2, v_3$.

The matrix representation of $M$ with respect to this basis is, denoted by $D$:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{pmatrix}
\]

Note $D^{100}$ is of the following form, thus has characteristic polynomial $(x - 1)(x - 2^{100})(x - 3^{100})$

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2^{100} & 0 \\
0 & 0 & 3^{100}
\end{pmatrix}
\]

Denote $A$ to be the change of basis matrix from $\{v_1, v_2, v_3\}$ to the standard basis, we then have: $M = ADA^{-1}$. 

\[5\]
On the other hand,

\[
\det(xM^{100} - Id) = \det A \det(xM^{100} - Id) \det A^{-1}
\]
\[
= \det(xAM^{100}A^{-1} - AA^{-1})
\]
\[
= \det(x(AMA^{-1})^{100} - Id)
\]
\[
= \det(xD^{100} - Id)
\]

Hence \(M^{100}\) and \(D^{100}\) have the same characteristic polynomial, which is \((x-1)(x-2^{100})(x-3^{100})\). 

\(\square\)