1 Recap

For any finite dimensional vector space $V$ over a field $F$ and $A \in \mathcal{L}(V, V)$, we defined

$$\det A = \frac{\Lambda(Av_1, \ldots, vAv_n)}{\Lambda(v_1, \ldots, v_n)},$$

where $\Lambda$ is a nonzero alternating $n$-form and $v_1, \ldots, v_n$ is a basis. We proved that this is independent of the choice of $\Lambda$ and $v_i$. We further proved that

1. $\det AB = (\det A)(\det B)$
2. $\det A \neq 0 \iff A$ is an isomorphism

If $M$ is an $n \times n$ matrix, we write $\det M$ for the determinant of the map on $F^n \to F^n$ given by $x \mapsto Mx$ (think of $x$ as a column vector). We saw last time that (in a special case),

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

**Proposition 1.** Suppose $T: V \to V$ has matrix $M$ with respect to bases $(e_1, e_2, \ldots, e_n)$ (so if $M = (a_{ij})$, then $T(e_i) = \sum_{j=1}^n a_{ji}e_j$). Then, $\det T = \det M$, where $\det T$ is as a linear operator $V \to V$, and $\det M$ is as defined above, i.e. as the determinant of a linear map on $F^n \to F^n$.

**Proof.** Exercise. \qed
2 Computing Determinants

The homework this week explores how to practically compute the determinant of a matrix. For any matrix $M$, the determinant $\det M$ is left unchanged by “row operations”, e.g. adding a multiple of one row to another (and similarly, column operations). We can use these to reduce $M$, for example, to a triangular form like

$$
\begin{bmatrix}
  a_{11} & ? & ? & ? \\
  0 & a_{22} & ? & ? \\
  0 & 0 & a_{33} & ? \\
  0 & 0 & 0 & a_{44} \\
\end{bmatrix}
$$

The determinant of a triangular matrix is the product of the diagonal entries; $a_{11}a_{22}a_{33}a_{44}$ above.

Another approach is to use cofactor expansion: Suppose

$$
M = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn} \\
\end{bmatrix}.
$$

Then, write $M^{(ij)}$ for the $n-1 \times n-1$ matrix obtained by deleting the row and column through $(i, j)$. For example, if

$$
M = \begin{bmatrix}
  1 & 2 & 3 \\
  3 & 2 & 1 \\
  -1 & 1 & 1 \\
\end{bmatrix}, \quad M^{(11)} = \begin{bmatrix}
  2 & 1 \\
  1 & 1 \\
\end{bmatrix}, \quad M^{(12)} = \begin{bmatrix}
  3 & 1 \\
  -1 & 1 \\
\end{bmatrix}, \quad M^{(13)} = \begin{bmatrix}
  3 & 2 \\
  -1 & 1 \\
\end{bmatrix}.
$$

We can compute the determinant of any matrix $M$ as

$$
\det M = a_{11} \det M^{(11)} - a_{12} \det M^{(12)} + a_{13} \det M^{(13)} - \cdots + (-1)^{n+1} a_{1n} \det M^{(1n)}.
$$

In our example,

$$
\det M = 1 \det \begin{bmatrix}
  2 & 1 \\
  1 & 1 \\
\end{bmatrix} - 2 \det \begin{bmatrix}
  3 & 1 \\
  -1 & 1 \\
\end{bmatrix} + 3 \det \begin{bmatrix}
  3 & 2 \\
  -1 & 1 \\
\end{bmatrix} = 1 - 8 + 15 = 8.
$$

**Proposition 2.** If you swap two rows (or two columns), the sign of the determinant changes.

**Proof.** Exercise. Use the alternating property of alternating $n$-forms.

For example,

$$
\det \begin{bmatrix}
  3 & 2 & 1 \\
  1 & 2 & 3 \\
  -1 & 1 & 1 \\
\end{bmatrix} = -8.
$$
Proposition 3. We can compute the determinant of a matrix $M$ as
\[
\det M = a_{11} \det M^{(11)} - a_{12} \det M^{(12)} + a_{13} \det M^{(13)} - \cdots + (-1)^{n+1} a_{1n} \det M^{(1n)}.
\]

Proof. We will actually prove a symmetric theorem for expanding a single column.

$M$ defines a linear map $F^n \to F^n$. Let $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, \ldots)$, etc.; i.e. let $e_i$ be the “standard basis” of $F^n$. Let $\Lambda$ be a nonzero alternating $n$-form on $F^n$. Then,
\[
\det M = \frac{\Lambda(Me_1, Me_2, \ldots, Me_n)}{\Lambda(e_1, \ldots, e_n)}.
\]

Recall that
\[
Me_1 = a_{11}e_1 + a_{21}e_2 + a_{31}e_3 + \cdots + a_{n1}e_n
\]
\[
Me_2 = a_{12}e_1 + a_{22}e_2 + a_{32}e_3 + \cdots + a_{n2}e_n
\]
\[
\vdots
\]
\[
Me_n = a_{1n}e_1 + a_{2n}e_2 + a_{3n}e_3 + \cdots + a_{nn}e_n
\]

So, by linearity in the first entry,
\[
\det M = \sum_{i=1}^{n} a_{i1} \frac{\Lambda(e_i, Me_2, \ldots, Me_n)}{\Lambda(e_1, e_2, \ldots, e_n)}.
\]

For example, the $i = 1$ entry is
\[
a_{11} \frac{\Lambda(e_1, Me_2, \ldots, Me_n)}{\Lambda(e_1, e_2, \ldots, e_n)}.
\]

We want to show that this is equal to $a_{11} \det M^{(11)}$. Define an alternating $(n - 1)$-form $\Lambda'$ on span$(e_2, \ldots, e_n)$ by the rule
\[
\Lambda'(v_1, \ldots, v_{n-1}) = \Lambda(e_1, v_1, \ldots, v_{n-1}).
\]

Note that this satisfies the axioms (multilinear, alternating) because $\Lambda$ does; and is nonzero since $\Lambda'(e_2, \ldots, e_n) = \Lambda(e_1, \ldots, e_n) \neq 0$. We want to show that
\[
\frac{\Lambda(e_1, Me_2, \ldots, Me_n)}{\Lambda(e_1, e_2, \ldots, e_n)} = \det M^{(11)}.
\]

Ideally, we’d like to say that
\[
\Lambda(e_1, Me_2, \ldots, Me_n) = \Lambda'(Me_2, \ldots, Me_n).
\]

Recall, though, that $Me_i$ is not in span$(e_1, \ldots, e_n)$; for example, $Me_2 = a_{12}e_1 + a_{22}e_2 + \cdots + a_{n2}e_n$.

However, by linearity (first in the second term, then in the rest), we see that
\[
\Lambda(e_1, Me_2, Me_3, \ldots, Me_n) = \Lambda(e_1, a_{12}e_1, Me_3, \ldots, Me_n) + \Lambda(e_1, Me_2 - a_{12}e_1, \ldots, Me_n)
\]
\[
= \Lambda(e_1, Me_2 - a_{12}e_1, \ldots, Me_n) \quad \text{since $\Lambda$ is alternating}
\]
\[
= \Lambda(e_1, Me_2 - a_{12}e_1, Me_3 - a_{13}e_1, \ldots, Me_n - a_{1n}e_n)
\]
\[
= \Lambda'(Me_2 - a_{12}e_1, Me_3 - a_{13}e_1, \ldots, Me_n - a_{1n}e_n).
\]


So,
\[
\frac{\Lambda(e_1, Me_2, \ldots, Me_n)}{\Lambda(e_1, e_2, \ldots, e_n)} = \frac{\Lambda'(Me_2, \ldots, Me_n)}{\Lambda'(e_2, \ldots, e_n)}.
\]
This is the determinant of the transformation \(T\) from \(\text{span}(e_2, \ldots, e_n)\) to \(\text{span}(e_2, \ldots, e_n)\) so that \(T(e_j) = Me_j - a_{1j}e_1\). The matrix of \(T\) with respect to basis \(e_2, \ldots, e_n\) is precisely \(M^{(11)}\). Hence, we indeed have
\[
\frac{\Lambda(e_1, Me_2, \ldots, Me_n)}{\Lambda(e_1, e_2, \ldots, e_n)} = \det M^{(11)}.
\]

Exercise: Check that the other terms are really given by \((-1)^{j+1}a_{j1}\det M^{(j1)}\).

\[
\square
\]

3 Reading

Read Axler chapters 4 and 5 for the next class. We’ll skip over chapter 4 (on polynomials) in lecture, and talk about chapter 5.