1 Recap

Last time, we discussed the Gram-Schmidt process.

For example, if we take $V$ to be the space of polynomials of degree $\leq N$ from $[-1,1]$ to $\mathbb{R}^n$ with inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$, and start with basis $e_1 = 1, e_2 = x, e_3 = x^2, \ldots$, then by the Gram-Schmidt process, we get the orthonormal basis

$$f_1 = \frac{e_1}{\|e_1\|} = \frac{1}{\sqrt{2}}$$

$$f'_2 = e_2 - \langle e_2, f_1 \rangle f_1$$

$$f_2 = \frac{f'_2}{\|f'_2\|} = x$$

$$f_3 = \alpha(3x^2 - 1)$$

$$f_4 = \beta(5x^3 - 3x^2)$$

(where $\alpha$ and $\beta$ are constants that scale the polynomials to the appropriate length). These $f_i$’s are called Legendre polynomials. They arise often in studying systems with spherical symmetry. $f_i(\cos \theta)$ are examples of “spherical harmonics”.

More generally, given any interval $[A,B]$ and a function $w(x) > 0$ for $x \in [A,B]$, we can apply the Gram-Schmidt process to polynomials on $[A,B]$ with respect to inner product

$$\langle f, g \rangle = \int_{A}^{B} w(x)f(x)g(x)dx.$$ 

This gives “orthogonal polynomials”; these arise very often (for various $A, B, w$) in math and physics.
2 The Adjoint of a Linear Transformation

We will now look at the adjoint (in the inner-product sense) for a linear transformation. A self-adjoint linear transformation has a basis of orthonormal eigenvectors $v_1, \ldots, v_n$.

Earlier, we defined for $T : V \to W$ the adjoint $\hat{T} : W^* \to V^*$. If $V$ and $W$ are inner product spaces, we can “reinterpret” the adjoint as a map $T^* : W \to V$. The motivation for this construction is something like the following: Earlier we saw that a bilinear pairing $X \times Y \to F$ (where $X, Y$ are vector spaces over $F$) induces maps $X \to Y^*$ and $Y \to X^*$. In the case that $F = \mathbb{R}$, then an inner product on $V$ — which gives a bilinear map on $V \times V \to \mathbb{R}$ — gives an isomorphism of $V$ and $V^*$. Roughly, an inner product gives a way to equate $V$ and $V^*$.

**Definition 1** (Adjoint). If $V$ and $W$ are finite dimensional inner product spaces and $T : V \to W$ is a linear map, then the **adjoint** $T^*$ is the linear transformation $T^* : W \to V$ satisfying for all $v \in V, w \in W$,

$$\langle T(v), w \rangle = \langle v, T^*(w) \rangle.$$

**Lemma 2.1** (Representation Theorem). If $V$ is a finite dimensional inner product space and $\ell : V \to F$ ($F = \mathbb{R}$ or $\mathbb{C}$) is a linear functional, then there exists a unique $w \in V$ so that $\ell(v) = \langle v, w \rangle$ for all $v \in V$.

*Proof.* Proof left as an exercise (use an orthonormal basis). \(\square\)

This theorem is called a “representational theorem” because it shows that you can represent a linear functional $\ell \in V^*$ by a vector $w \in V$.

Why does $T^*$ (as in the definition of an adjoint) exist? For any $w \in W$, consider $\langle T(v), w \rangle$ as a function of $v \in V$. It is linear in $v$. By the lemma, there exists some $y \in V$ so that $\langle T(v), w \rangle = \langle v, y \rangle$. Now we define $T^*(w) = y$. This gives a function $W \to V$; we need only to check that it is linear.

Properties of $T^*$:

1. If $e_1$ is an orthonormal basis for $V$ and $f_j$ is an orthonormal basis for $W$, then the matrix of $T$ with respect to $e_i, f_j$ is the **conjugate transpose** of the matrix of $T^*$ with respect to $f_j, e_i$.

   For example, if $V = \mathbb{C}^2, W = \mathbb{C}^2$, the inner product is $\langle(z_1, w_1), (z_2, w_2)\rangle = z_1 \overline{w_2} + w_1 \overline{w_2}$, and $T$ is defined by $\begin{bmatrix} 1 & i \\ 0 & i \end{bmatrix}$, then $T^*$ is defined by $\begin{bmatrix} 1 & 0 \\ -i & -i \end{bmatrix}$.

2. If $T_1, T_2 \in \mathcal{L}(V, W)$ and $\lambda_1, \lambda_2 \in F$, then

   $$(\lambda_1 T_1 + \lambda_2 T_2)^* = \overline{\lambda_1} T_1^* + \overline{\lambda_2} T_2^*.$$

3. $(T^*)^* = T$ (follows directly from the definition).
4. The range (image) of $T$ is the perpendicular space of the nullspace of its adjoint, and vice-versa:

\[
\text{image}(T) = \text{null}(T^*)^\perp \\
\text{null}(T) = \text{image}(T^*)^\perp
\]

and similarly exchanging $T$ and $T^*$.

Most of the proofs are very similar to previous ones about $\hat{T}$ (see Axler, end of Chapter 6). For example,

**Proposition 1.** $\text{image}(T) = \text{null}(T^*)^\perp$.

**Proof.** Note that both $\text{image}(T), \text{null}(T^*) \subseteq W$.

We will actually show that $\text{image}(T)^\perp = \text{null}(T^*)$; this implies the desired result, since $(\text{image}(T)^\perp)^\perp = \text{image}(T)$.

Recall that $w \in \text{image}(T)^\perp$ if and only if $\langle T(v), w \rangle = 0$ for all $v \in V$. By the definition of the adjoint, this is if and only if $\langle v, T^*(w) \rangle = 0$ for all $v \in V$, i.e. $T^*(w) = 0$, so $w \in \text{null}(T^*)$.

Note that at the last step, we used the fact that if $v_0 \in V$ satisfies $\langle v, v_0 \rangle = 0$ for all $v$, then $v_0 = 0$. Indeed, this implies that $\langle v_0, v_0 \rangle = 0$, and hence by an axiom of inner products, $v_0 = 0$.

\[\square\]

## 3 Self-Adjoint

Recall that we want:

**Theorem 3.1.** If $T : V \to V$ (where $V$ is a finite dimensional inner product space over $F$) so that $T = T^*$ ("self-adjoint"), then there is an orthonormal basis of eigenvectors and all eigenvalues are real.

**Proof.** Why are all eigenvalues real? Given $v$ an eigenvector with eigenvalue $\lambda$, i.e. $T(v) = \lambda v$, we can consider

\[
\lambda \langle v, v \rangle = \langle \lambda v, v \rangle = \langle T(v), v \rangle = \langle v, T(v) \rangle = \langle v, \lambda v \rangle = \overline{\lambda} \langle v, v \rangle.
\]

Note that $\langle T(v), v \rangle = \langle v, T(v) \rangle$ because $T$ is self-adjoint. From the above, we see that $\lambda = \overline{\lambda}$, i.e. $\lambda$ is real.

Now, let’s prove the rest of the theorem for $F = \mathbb{C}$. We know that there exists an eigenvector $v \in V$. Let $U$ be $\text{span}(v)^\perp$. Then, $T$ preserves $U$: if $u \in U$, then $T(u) \in U$. Now, by induction on the dimension, we can find an orthonormal basis of eigenvectors on $U$.

\[\square\]

This is perhaps the most important result in the course.