Let $V$ and $W$ be a finite-dimensional complex Euclidean vector spaces.

**Theorem 1.** Let $T : V \to W$ be a linear map. Then there is a unique linear map $S : W \to V$ such that

$$(Tv, w) = (v, Sw) \text{ for all } v \in V \text{ and } w \in W.$$  

If we use orthonormal bases for $V$ and for $W$ to represent $T$ and $S$ by matrices, then

$$S_{ij} = T_{ji}.$$  

**Definition.** The map $S$ is called the adjoint of $T$ and is denoted $T^*$. 

**Proof.** Let $e_1, \ldots, e_n$ and $f_1, \ldots, f_m$ be orthonormal bases for $V$ and for $W$, respectively. Then

$$(Tv, w) = (T(\sum v_i e_i), \sum w_j f_j)$$

$$= \sum_{i,j} v_i \overline{w_j} (Te_i, f_j)$$

$$= \sum_{i,j} v_i \overline{w_j} \text{(the } f_j \text{ component of } Te_i)$$

$$= \sum_{i,j} v_i \overline{w_j} T_{ji}. \quad (2)$$

Almost exactly the same calculation shows that

$$(v, Sw) = \sum_{i,j} v_i \overline{w_j} \overline{S_{ij}}. \quad (3)$$

Thus if we let $S_{ij}$ is the conjugate of $T_{ji}$ (for all $i$ and $j$), then (2) and (3) will be equal for all $v$ and $w$. This establishes the existence of a map $S$ with property (1).

Conversely, if $S$ has the property (1), then (2) and (3) will be equal for all $v$ and $w$. In particular, they will be equal when $v = e_i$ and $w = f_j$, in which case case (2) is $T_{ji}$ and (3) is the conjugate of $S_{ij}$. This proves uniqueness. □

**Theorem 2.** Let $V$ be a finite-dimensional complex Euclidean vector space. A linear map $T : V \to V$ has an orthonormal basis of eigenvectors if and only if $T^* T = TT^*$.

**Remark.** A linear transformation $T : V \to V$ such that $T^* T = TT^*$ is called normal.
Lemma 1. If $T : V \to V$ is normal, then $|T^*v| = |Tv|$ for all $v$.

Proof of lemma 1.

$$|T^*v|^2 = (T^*v, T^*v) = (TT^*v, v) = (T^*Tv, v) = (Tv, Tv) = |Tv|^2$$

□

Lemma 2. Suppose $T : V \to V$ is normal. If $v$ is an eigenvector of $T$ with eigenvalue $\lambda$, then it is also an eigenvector of $T^*$, with eigenvalue $\bar{\lambda}$. In other words,

$$Tv = \lambda v \implies T^*v = \bar{\lambda}v.$$

Proof. If $T$ is normal, then so is $(T - \lambda I)$. Thus by lemma 1 applied to $T - \lambda I$,

$$|(T - \lambda I)v| = |(T - \lambda I)^*v| = |T^*v - \bar{\lambda}v|$$

Since $Tv = \lambda v$, the left side is 0. Therefore the right side is 0, which means that $T^*v = \bar{\lambda}v$. □

Proof of theorem 2. Any linear transformation of a complex vector space has at least one eigenvector. So let $v_1$ be a unit eigenvector for $T$.

Now suppose we have orthonormal eigenvectors $v_1, \ldots, v_k$ for $T$. If $k = \dim V$, we are done. If not, let $W$ be the set of all vectors that are orthogonal to $v_1, \ldots, v_k$:

$$W = \{x \in V : (x, v_i) = 0 \text{ for } i = 1, \ldots, k\}.$$

Then $W$ is an $n - k$ dimensional subspace (where $n = \dim V$.)

Claim: If $x$ is perpendicular to $v_i$, then so is $Tx$.

Proof of claim. If $x$ is perpendicular to $v_i$, then

$$(Tx, v_i) = (x, T^*v_i) = (x, \bar{\lambda_i}v_i) = \lambda(x, v_i) = 0.$$

Thus $(Tx, v_i) = 0$, so $Tx$ is perpendicular to $v_i$. This proves the claim.

Consequently, if $x$ is perpendicular to all the $v_i$’s, then so is $Tx$. In other words,

$$x \in W \implies Tx \in W.$$

Now since $T$ maps $W$ into itself, $W$ contains a unit eigenvector $v_{k+1}$.

Continuing in this way, we get an orthonormal basis of eigenvectors. □
Special Classes of Normal Transformations

Note that if $T^* = T$, or if $T^* = -T$, or if $T^* = T^{-1}$, then $T$ is normal. These conditions define the following classes of normal operators:

1. The eigenvalues of a normal operator $T$ are real if and only if $T^* = T$. Such a $T$ is called **self-adjoint** or **Hermitian**.

2. The eigenvalues of a normal operator $T$ are imaginary if and only if $T^* = -T$. Such a $T$ is called **skew-adjoint** or **skew-Hermitian**.

3. The eigenvalues of a normal operator $T$ are unit complex numbers (i.e., have norms 1) if and only if $T^* = T^{-1}$. Such a $T$ is called **unitary**.

Unitary operators have other important characterizations:

**Proposition.** The $T : V \rightarrow V$ be an operator on the finite dimensional complex Euclidean space $V$. The following are equivalent:

1. $T$ is unitary.
2. $(Tu, Tv) = (u, v)$ for all vectors $u$ and $v$ in $V$.
3. $|Tv| = |v|$ for all vectors $v$ in $V$.

**Proof.** Exercise.