MATH 52H HOMEWORK 7 (Due Friday, February 22)

Note: there are some new lecture notes on the website (go to the course homepage.) In particular, the lecture notes about normal operators are relevant for problem 3.

Check back later for the text problems.

1. Let $W$ be the set of vectors in $\mathbb{R}^5$ that are perpendicular to $(1, 1, 1, 1, 1)$, $(2, 2, 3, 3, 4)$, and $(3, 3, 4, 4, 5)$. Find a basis for $W$.

2. Let $u_1, \ldots, u_k$ be orthonormal vectors in $\mathbb{R}^n$. Let $W$ be the subspace that they span. Let $U$ be the $n \times k$ matrix whose $j$th column is $u_j$. Prove that

$$UU^t$$

is the matrix for orthogonal projection to $W$. In other words, prove that if $v \in \mathbb{R}^n$, then $UU^tv$ is the vector in $W$ closest to $v$.

3. Suppose $A$ and $B$ are linear operators on the finite dimensional complex Euclidean space $V$. Suppose also that $A$ and $B$ commute: $AB = BA$.

(a). Prove that if $v$ is an eigenvector of $B$, then so is $Av$ (unless $Av = 0$.)

(b). Prove that $A$ and $B$ have a common eigenvector. That is, prove that there is a nonzero vector $v$ such that $Av = \lambda v$ and $Bv = \mu v$ for suitable scalars $\lambda$ and $\mu$.

(c). Show that if $A$ and $B$ are normal and if $AB = BA$, then there is an orthonormal basis $v_1, \ldots, v_n$ of $V$ such that each $v_i$ is an eigenvector of $A$ and of $B$.

(d). Prove the converse of part (c): If $A$ and $B$ are normal operators on $V$ and if there is an orthonormal basis of common eigenvectors, then $AB = BA$.

4. Suppose $A$ is an $n \times n$ matrix all of whose entries are rational numbers. Suppose one of the eigenvalues of $A$ is also a rational number. Prove that $A$ has an eigenvector all of whose entries are rational numbers.

5. Suppose $A$ is an $n \times n$ matrix with a basis of eigenvectors. Suppose that each eigenvalue $\lambda_i$ is $< 1$ in absolute value: $|\lambda_i| < 1$. Prove that $A^n x \to 0$ as $n \to \infty$ for every $x$. 

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