There are several efficient methods for finding a basis for a subspace. To discuss them, it is useful to have some terminology. The “column space” of a \( k \times n \) matrix \( A \) is the set of all linear combinations of the columns. The column space is the same as the image since:

\[
Ax = (\text{column 1 of } A)x_1 + \ldots + (\text{column } n \text{ of } A)x_n.
\]

The row space is the set of all linear combinations of the rows.

A “row operation” on \( A \) is one of the following:

1. adding a multiple of one row to another row,
2. multiplying a row by a nonzero scalar, and
3. switching two rows.

Recall that for each row operation there is a corresponding invertible \( k \times k \) matrix \( E \). In particular, if performing the row operation on \( A \) produces \( A' \), then \( A' = EA \). Of course if we perform a sequence of row operations on \( A \) to get a new matrix \( B \), then

\[
B = E_pE_{p-1}\ldots E_2E_1A = MA,
\]

where \( E_i \) corresponds to the \( i \)th row operation, and \( M \) is the product of the \( E_i \)'s. Thus we have shown:

**Proposition 1.** If the \( k \times n \) matrix \( B \) is obtained from \( A \) by a sequence of row operations, then

\[
B = MA
\]

for some invertible \( k \times k \) matrix \( M \).

Suppose we want to find a basis for \( W = L(\mathbf{v}_1, \ldots, \mathbf{v}_k) \). We can either:

1. Make a matrix \( A \) whose rows are the \( \mathbf{v}_i \)'s and find a basis for the row space of \( A \) (which is \( W \)), or
2. make a matrix \( A \) whose columns are the \( \mathbf{v}_i \)'s and find a basis for the column space of \( A \).

Such bases may be found quickly with row operations.

**To find a basis for the row space of \( A \):** Perform row operations to get a matrix \( B \) in reduced row echelon form (rref). The rows of \( B \) are then a basis for \( W \).
(Recall that in each nonzero row of a matrix, the first (or leftmost) position with a nonzero entry is called a \textbf{pivot}. A matrix in rref has at most one pivot in each column.)

Why does this work? Note that row operations do not change the row space. Thus the row space of $B$ is the same as the row space of $A$ which is the same as $W$.

The reader should as an exercise prove that rref implies that the nonzero row of $B$ form a basis for its row space (and hence for $W$.)

\textbf{To find a basis for the column space of $A$:} Perform row operations to get a matrix $B$ in rref. The columns of $A$ corresponding to pivots in $B$ form a basis for the column space of $A$.

Why does this work? By the discussion above, we know there is an invertible $k \times k$ matrix $M$ such that

\[ B = MA. \]

Let the columns of $A$ be $A_1, \ldots, A_n$ and the columns of $B$ be $B_1, \ldots, B_n$. Then $B_j = MA_j$ for each $j$.

Claim 1: $A_j$ is a linear combination of $A_1, \ldots, A_{j-1}$ if and only $B_j$ is a combination of $B_1, \ldots, B_{j-1}$.

Proof: Suppose

(i) \[ A_j = c_1 A_1 + \cdots + c_{j-1} A_{j-1}. \]

Multiplying both sides by $M$ gives

(ii) \[ B_j = c_1 B_1 + \cdots + c_{j-1} B_{j-1}. \]

Likewise, if we assume (ii), then multiplying by $M^{-1}$ gives (i). This proves claim 1.

Claim 2: $B_j$ is a combination of $B_1, \ldots, B_{j-1}$ if and only if column $j$ of $B$ contains no pivot.

Proof is left to the reader.

Putting claims 1 and 2 together, we see that this method picks out the $A_j$’s which are not combinations of preceding $A_i$’s. We already know that those $A_j$’s do form a basis.

***

\textbf{Warning:} Though $A$ and $B$ have the same row spaces, their column spaces will generally be different.
Find bases for the row space and for the column space of

\[
A = \begin{bmatrix}
1 & 2 & 1 & 3 & -1 \\
1 & 2 & 2 & 2 & 3 \\
1 & 2 & 3 & 1 & 7
\end{bmatrix}.
\]

**Solution:** Subtract row 1 from rows 2 and 3:

\[
\begin{bmatrix}
1 & 2 & 1 & 3 & -1 \\
0 & 0 & 1 & -1 & 4 \\
0 & 0 & 2 & -2 & 8
\end{bmatrix}.
\]

Now subtract row 2 from row 1 and twice row 2 from row 3:

\[
B = \begin{bmatrix}
1 & 2 & 0 & 4 & -5 \\
0 & 0 & 1 & -1 & 4 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

The pivots of \(B\) (which is in rref) are in columns 1 and 3. Thus columns 1 and 3 of \(A\) (namely \((1,1,1)\) and \((1,2,3)\)) form a basis for the column space of \(A\).

The nonzero rows of \(B\), namely \((1, 2, 0, 4, -5)\) and \((0, 0, 1, -1, 4)\), form a basis for the row space of \(A\).