Math 51h Homework 9 (due Friday, November 30, 2001)

1. Let $C$ be the portion of the curve:

$$x + \sqrt{xy} + 3y = 15$$

starting at $A = (1, 4)$ and ending at $B = (9, 1)$. Find $\int_C y^2 \, dx + (2xy + 3y^2) \, dy$.

**Solution:** Note that $\frac{\partial}{\partial y} (y^2) = 2y = \frac{\partial}{\partial x} (2xy + 3y^2)$ in the convex set $\mathbb{R}^2$. Thus $F = (y^2)i + (2xy + 3y^2)j$ is the gradient of some function $\phi$:

$$\frac{\partial \phi}{\partial x} = y^2 \quad \frac{\partial \phi}{\partial y} = 2xy + 3y^2$$

Integrating the first equation, we see that $\phi = xy^2 + C(y)$. Substituting into the second equation gives

$$\frac{\partial}{\partial y} (xy^2 + C(y)) = 2xy + 3y^2$$

or $2xy + C'(y) = 2xy + 3y^2$ or $C'(y) = 3y^2$ or $C(y) = y^3 + K$. We may as well let $K = 0$. Thus $\phi = xy^2 + y^3$. Now

$$\int_C y^2 \, dx + (2xy + y^3) \, dy = \phi(B) - \phi(A) = (9 \cdot 1^2 + 3 \cdot 1^3) - (1 \cdot 4^2 + 4^3) = -68$$

2(a). Let $S$ be a nonempty closed set in $\mathbb{R}^n$. Let $a$ be a point in $\mathbb{R}^n$. Prove that there is a point $x$ in $S$ closest to $a$. (There may be more than one such point.) **Note:** $S$ may not be bounded.

**Solution:** Since $S$ is nonempty, there is a point $y$ in $S$. Let $R = |y - a|$. Let

$$S^* = \{x \in S : |x - a| \leq R\}$$

Let $f(x) = |x - a|$. Then $S^*$ is a closed bounded set and $f$ is a continuous function, so $f$ has a minimum on $S^*$ at some point $x_0$. That is: $f(x_0) \leq f(x)$ for every $x \in S^*$. But if $x$ is in $S$ but not in $S^*$, then $f(x) > R \geq f(x_0)$. Thus $f(x_0) \leq f(x)$ for every $x \in S$. □

2(b). Suppose the set $S$ in part (a) is not all of $\mathbb{R}^n$. Prove that the boundary of $S$ is not empty.
Solution: Since $S$ is not all of $\mathbb{R}^n$, there is a point $a \in \mathbb{R}^n$ not in $S$. Let $p$ be a point in $S$ closest to $a$ (this exists by part (a)). Note that every point in the line segment joining $a$ to $p$ is closer to $a$ than $p$ is. Thus none of the points of the line segment (except $p$) belong to $S$. Now any ball centered at $p$ must contain points in this segment. Thus $p$ is not the interior of $S$. Also $p$ itself is in $S$, so $p$ cannot be an exterior point. Thus $p$ is a boundary point of $S$. □

3. Suppose $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a continuously differentiable vectorfield and that $D_iF_j = D_jF_i$ for all $i$ and $j$ and at all points of $\mathbb{R}^3$. Let $S$ be a sphere in $\mathbb{R}^3$. Prove that there is some point $x$ in $S$ such that $F(x)$ is normal to $S$ at $x$.

Proof: Since $D_iF = D_jF_i$ throughout $\mathbb{R}^3$ and since $\mathbb{R}^3$ is a convex set, $F$ must be the gradient of some function $u$. If $S$ has center $a$ and radius $R$, then $S$ is the level set of a smooth function $g$, namely

$$g(x) = |x - a|^2.$$

(Of course $S$ is the set of all points where $g = R^2$.)

By the extreme value theorem, $u$ has a maximum and minimum value subject to the constraint $g = R^2$. Let $x \in S$ be a point where the maximum (say) occurs. By the lagrange multiplier theorem, $\nabla u(x)$ and $\nabla g(x)$ are linearly dependent. Since $\nabla g(x)$ is not zero, this means

$$(*) \quad \nabla u(x) = \lambda \nabla g(x),$$

where $\lambda$ is a real number. Now $\nabla g(x)$ is normal to $S$ at $x$ (since $S$ is a level set of $g$), so $(*)$ implies that $\nabla u(x)$ is normal to $S$ at $x$. But $F(x) = \nabla u(x)$, so we are done. □