MATH 51H HOMEWORK 5 SOLUTIONS

Extra problems:

1. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuous. Let $S = \{a \in \mathbb{R}^n : f(a) > 0\}$. Prove that $S$ is open.

**Proof.** Let $a \in S$. We must show that there is a ball $B(a, r)$ around $a$ that is also contained in $S$. Since $a \in S$, $f(a) > 0$.

Let $\epsilon = f(a)$. Then $\epsilon > 0$, so by continuity of $f$, there is a $\delta > 0$ such that

$$|x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

Thus if $x \in B(a, \delta)$, then $|x - a| < \delta$, so

$$|f(x) - f(a)| < \epsilon = f(a)$$

that is,

$$-f(a) < f(x) - f(a) < f(a)$$

so (adding $f(a)$ to the inequality)

$$0 < f(x)$$

so $x \in S$.

We have shown $x \in B(a, \delta) \implies x \in S$. Thus $B(a, \delta) \subset S$, as was to be proved. □

2. Let $f(x, y) = e^{x \sin y} + \sqrt{x + y}$.

2(a). Find the derivative $Df(x, y)$, and use your answer to do parts (b) and (c).

2(b). Find $f(4, 0)$, and estimate $f(4.01, 0.02)$.

2(c). Find a $y$ that very nearly solves $f(4.008, y) = 3.019$.

**SOLUTION.**

$$Df(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ e^{x \sin y} \sin y + \frac{1}{2}(x + y)^{-1/2} & xe^{x \sin y} \cos y + \frac{1}{2}(x + y)^{-1/2} \end{bmatrix}$$

Thus $Df(4, 0) = \begin{bmatrix} \frac{1}{4} & 4 \frac{1}{3} \end{bmatrix}$. Also $f(4, 0) = 3$. Thus

$$f(4.01, .02) = f((4, 0) + (.01, .02))$$

$$\approx f(4, 0) + Df(4, 0) \begin{bmatrix} .01 \\ .02 \end{bmatrix}$$

$$= 3 + \begin{bmatrix} \frac{1}{4} & 4 \frac{1}{3} \end{bmatrix} \begin{bmatrix} .01 \\ .02 \end{bmatrix}$$

$$= 3 + .0025 + .085 = 3.0875$$

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(The true answer, according to my computer, is 3.091. If we do this with \( f(0.001, .002), \) we get an estimate of 3.0088 and a “true” answer of 3.0088.)

For small \( y, \)
\[
f(4.008, y) \approx f(4, 0) + Df(4, 0) \begin{bmatrix} .008 \\ y \end{bmatrix}
= 3 + \left[ \frac{1}{4} \ 4 \frac{1}{4} \right] \begin{bmatrix} .008 \\ y \end{bmatrix}
= 3 + .002 + (4 \frac{1}{4})y
\]

We want this to equal 3.019, so we solve 3.002\((4 \frac{1}{4})y = 3.019\) for \( y \) and get \( y = .004. \) (This “nearly” solves the equation: according to my computer, \( f(4.008, .004) = 3.0192. \))

\(3(a). \) Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \) where
\[
f \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x^3 + xy + y^4 \\ x^3 + x + y + y^3 \end{bmatrix}.
\]
Find \( Df \left( \begin{bmatrix} x \\ y \end{bmatrix} \right). \)

What is \( Df \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)? \) What is \( f \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)? \)

\(3(b). \) Using your answers to part (a), find an approximate solution to the simultaneous equations
\[
x^3 + xy + y^4 = 3.2 \\
x^3 + x + y + y^3 = 4.3
\]

**SOLUTION.** Let \( x = 1 + u \) and \( y = 1 + v. \) We want to solve \( f(1 + u, 1 + v) = (3.2, 4.3). \) But
\[
f(1 + u, 1 + v) \approx f(1, 1) + Df(1, 1) \begin{bmatrix} u \\ v \end{bmatrix}
= \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}
\]

That is, we want
\[
\begin{bmatrix} 4 & 5 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} .2 \\ .3 \end{bmatrix}
\]
or \(4u + 5v = .2 \) and \(4u + 4v = .3. \)

Solving these gives \( v = -.1 \) and \( u = .175, \) or \( x = 1.175 \) and \( y = .9. \) (In fact \( f(1.175, .9) = (3.3358, 4.4262). \))

\(4(a). \) Let \( S \) be all of \( \mathbb{R}^N \) except for one point. Prove that \( S \) is open.

**SOLUTION.** Let \( S \) be \( \mathbb{R}^N \setminus \{a\} \) (that is, all of \( \mathbb{R}^N \) except for the point \( a.\)) Let \( x \) be a point in \( S. \) Then \( x \neq a. \) Let \( r = \frac{|x-a|}{2}. \) Then \( B(x, r) \subset S. \) Thus \( x \) is an
interior point of $S$. Since $x$ was arbitrary, this means every point in $S$ is an interior point, so $S$ is open. □

Remark. We could have let $r = |x - a|$. 

4(b). Let $T$ be all of $\mathbb{R}^N$ except for a finite set of points. In other words, $T = \mathbb{R}^N \setminus \{x_1, \ldots, x_k\}$ for some points $x_1, \ldots, x_k$. Prove that $T$ is open.

**SOLUTION.** Let $y$ be a point in $T$. Then $y$ is not equal to any of $x_1, \ldots, x_k$, so the numbers $|y - x_i| (i = 1, \ldots, k)$ are all positive. Let $r$ be the smallest of those $k$ numbers. Then the ball $B(y, r) \subset T$, so $y$ is an interior point of $T$. Since $y$ was an arbitrary point in $T$, this means $T$ is open. □

4(c). Give an example of an infinite set $Z = \{x_1, x_2, x_3, \ldots\}$ of points such that $\mathbb{R}^N \setminus Z$ is not open.

**SOLUTION.** Let $W = \mathbb{R}^N \setminus \{x_1, x_2, \ldots\}$. We want for $W$ not to be open. That means we need a point $y$ in $W$ that is not an interior point. That means a point $y$, different from all the $x_i$’s, such that every ball around $y$ contains some of the $x_i$’s. For example, we could let $y$ be the origin, and $x_i$ be a sequence of nonzero points that converges to the origin. For example, let

$$x_i = \frac{1}{i} e_1$$