MATH 51h HOMEWORK 3 SOLUTIONS

1. Let \( X = B - A \) and \( Y = C - A \) and \( \alpha \) be the angle at vertex \( A \). Then

\[
\text{area}(\triangle ABC) = .5 \times \text{base} \times \text{height} = .5 \times |X|(|Y| \sin \alpha) = .5 \times |X||Y|\sqrt{1 - \cos^2 \alpha}
\]

\[
= .5 \times \sqrt{|X|^2|Y|^2 - |X|^2|Y|^2 \cos^2 \alpha} = .5 \sqrt{|X|^2|Y|^2 - (X \cdot Y)^2}
\]

In this problem \( X = (1, 3, 4, 5) - (1, 2, 3, 4) = (0, 1, 1, 1) \) and \( Y = (3, 0, 4, 0) - (1, 2, 3, 4) = (2, -2, 1, -4) \), so \( |X|^2 = 1^2 + 1^2 + 1^2 = 3 \), \( |Y|^2 = 2^2 + (-2)^2 + 1^2 + (-4)^2 = 25 \), and \( X \cdot Y = 0 \cdot 2 + 1 \cdot (-2) + 1 \cdot 1 + 1 \cdot (-4) = -5 \). Thus the area is .5 \sqrt{3 \cdot 25 - (-5)^2} = .5 \sqrt{50} = 5/\sqrt{2}.

2. (a) Recall that a parametric equation for the plane is \( \vec{A} + s(A_2 - \vec{A}) + t(A_3 - \vec{A}) \) or equivalently, \( X(s, t) = (1 - s - t)A_1 + sA_2 + tA_3 \). Thus if \( X \in \mathcal{P} \) then there are \( s \) and \( t \) such that \( X = (1 - s - t)A_1 + sA_2 + tA_3 \). Now we simply let \( a_1 = 1 - s - t, a_2 = s, \) and \( a_3 = t \).

Conversely, suppose \( X = a_1A_1 + a_2A_2 + a_3A_3 \) with \( \sum a_i = 1 \). Then \( X = (1 - s - t)A_1 + sA_2 + tA_3 \), where \( s = a_2 \) and \( t = a_3 \).

(b) Suppose \( X = \sum a_iA_i = \sum_i b_iA_i \), where \( \sum a_i = \sum b_i = 1 \). Then \( a_1 = 1 - a_2 - a_3 \) and \( b_1 = 1 - b_2 - b_3 \), so

\[
(1 - a_2 - a_3)A_1 + a_2A_2 + a_3A_3 = (1 - b_2 - b_3)A_1 + b_2A_2 + b_3A_3
\]

or

\[
A_1 + a_2(A_2 - A_1) + a_3(A_3 - A_1) = A_1 + b_2(A_2 - A_1) + b_3(A_3 - A_1)
\]

so

\[
a_2(A_2 - A_1) + a_3(A_3 - A_1) = b_2(A_2 - A_1) + b_3(A_3 - A_1)
\]

Since \( A_1, A_2, \) and \( A_3 \) are not collinear, \( A_2 - A_1 \) and \( A_3 - A_1 \) are linearly independent. Thus (*) implies that \( a_2 = b_2 \) and \( a_3 = b_3 \). \( \square \)

(c) Note

\[
X = \sum a_iA_i = A_1 + a_2(A_2 - A_1) + a_3(A_3 - A_1)
\]

From this it’s clear (draw the picture!) that \( X \) belongs to the region of \( \mathcal{P} \) in between the rays \( A_1A_2 \) and \( A_1A_3 \) if and only if \( a_2 \) and \( a_3 \) are both \( \geq 0 \). Similarly \( X \) is between the rays \( A_2A_3 \) and \( A_2A_1 \) if and only if \( a_1 \) and \( a_3 \) are both \( \geq 0 \). Putting these together, we see that \( X \) is in the triangle if and only if each \( a_i \geq 0 \).

(d) Since \( B_1 \) belongs to the line through \( A_2 \) and \( A_3 \), it must have the form

\[
B_1 = b_2A_2 + b_3A_3 \quad \text{where} \quad b_1 + b_2 = 1
\]

Now since \( X \) is on the line through \( A_1 \) and \( B \) it must have the form

\[
X = uA_1 + vB_1 = uA_1 + v(b_2A_2 + b_3A_3) \quad \text{where} \quad u + v = 1
\]

Now \( u + v b_2 + v b_3 = u + v(b_2 + b_3) = u + v = 1 \), so by part (b), we must have \( u = a_1, v b_2 = a_2, \) and \( v b_3 = a_3 \), or (since \( v = 1 - u = 1 - a_1 \))

\[
b_2 = a_2/(1 - a_1) = a_2/(a_2 + a_3) \quad b_3 = a_3/(1 - a_1) = a_3/(a_2 + a_3)
\]

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(e) Let \( X \) be the point of intersection of the three segments. By (a), (b), and (c), we can express \( X \) as
\[
X = \sum a_i A_i \quad \text{with} \quad \sum a_i = 1.
\]
By part (d),
\[
|A_2 - B_1| = \left| A_2 - \frac{a_2}{a_2 + a_3} A_2 - \frac{a_3}{a_2 + a_3} A_3 \right| = \frac{a_3}{a_2 + a_3} |A_2 - A_3|
\]
Likewise
\[
|B_1 - A_3| = \frac{a_2}{a_2 + a_3} |A_2 - A_3|
\]
so
\[
\frac{|A_2 - B_1|}{|B_1 - A_3|} = \frac{a_3}{a_2}
\]
Likewise
\[
\frac{|A_1 - B_3|}{|B_3 - A_2|} = \frac{a_2}{a_1} \quad \text{and} \quad \frac{|A_3 - B_2|}{|B_2 - A_1|} = \frac{a_1}{a_3}
\]
so
\[
\frac{|A_1 - B_3| |A_2 - B_1| |A_3 - B_2|}{|B_3 - A_2| |B_1 - A_3| |B_2 - A_1|} = \frac{a_2}{a_1} \frac{a_3}{a_2} = 1
\]
(f) Suppose the product is 1:
\[
(*) \quad \frac{|A_1 - B_3| |A_2 - B_1| |A_3 - B_2|}{|B_3 - A_2| |B_1 - A_3| |B_2 - A_1|} = 1
\]
Let \( X \) be the intersection of segment \( A_1B_1 \) and segment \( A_2B_2 \). Let \( B'_3 \) be the intersection of segment \( A_3X \) and segment \( A_1A_2 \). Then by (e), we know that
\[
\frac{|A_1 - B'_3| |A_2 - B_1| |A_3 - B_2|}{|B'_3 - A_2| |B_1 - A_3| |B_2 - A_1|} = 1
\]
Combining this with (*) gives
\[
\frac{|A_1 - B'_3|}{|B'_3 - A_2|} = \frac{|A_1 - B_3|}{|B_3 - A_2|}
\]
Since \( B_3 \) and \( B'_3 \) belong to segment \( A_1A_2 \), this implies \( B'_3 = B_3 \). \( \square \)

3.
\[
\int (f + g)^2 = \int (f^2 + 2fg + g^2) = \int f^2 + 2 \int fg + \int g^2 \leq \int f^2 + 2 \left( \int f^2 \right)^{1/2} \left( \int g^2 \right)^{1/2} + \int g^2 = \left( \int f^2 \right)^{1/2} + \left( \int g^2 \right)^{1/2}
\]
by the inequality \( \int fg \leq \left( \int f^2 \right)^{1/2} \left( \int g^2 \right)^{1/2} \) proved in last week’s homework.
As in the lecture notes on the web site about finding a basis, we make the given vectors into the rows of a matrix:

\[ A = \begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 5 & 7 & 10 \\
3 & 8 & 11 & 16 \\
1 & 3 & 4 & 7
\end{bmatrix}. \]

Now we simplify by adding or subtracting multiples of one row to other rows. Here we subtract 2 times row 1 from row 2, 3 times row 1 from row 3, and 1 times row 1 from row 4:

\[ \begin{bmatrix}
1 & 2 & 3 & 4 \\
0 & 1 & 1 & 2 \\
0 & 2 & 2 & 4 \\
0 & 1 & 1 & 3
\end{bmatrix}. \]

Next subtract 2 times row 2 from row 3, and 1 times row 2 from row 4:

\[ \begin{bmatrix}
1 & 2 & 3 & 4 \\
0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}. \]

This is not quite in “row echelon form”, so we switch rows 3 and 4:

\[ E = \begin{bmatrix}
1 & 2 & 3 & 4 \\
0 & 1 & 1 & 2 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}. \]

This matrix is in row echelon form. As explained in the notes, its row space (the subspace of \( \mathbb{R}^4 \) spanned by its rows) is the same as the row space of the original matrix \( A \). Also, the nonzero rows of \( E \) form a basis for this subspace. Thus \((1, 2, 3, 4), (0, 1, 1, 2), \) and \((0, 0, 0, 1)\) form a basis.

NOTE: Some people prefer to further simplify the matrix so that each column has at most one pivot in it. A row echelon matrix with this property is said to be in “reduced row echelon form”. To get a reduced row echelon form matrix from \( E \), first subtract 2 times row 3 from row 2 and 4 times row 3 from row 1:

\[ \begin{bmatrix}
1 & 2 & 3 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}. \]

Next, subtract 2 times row 2 from row 1:

\[ \tilde{E} = \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}. \]

This matrix is in reduced row echelon form. Its nonzero rows, namely \((1, 0, 1, 0), (0, 1, 1, 0), \) and \((0, 0, 0, 1)\) also form a basis for the subspace described in this problem.
Classify points in $\mathbb{Z}^3$ into types according to whether each component is even or odd. Thus for example $(5, 2, 8)$ has the type $(\text{odd, even, even})$. Let $A$ and $B$ be points in $\mathbb{Z}^3$. From the formula

$$\text{midpoint}(A, B) = A + B = \frac{A + B}{2} = \left( \frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{2}, \frac{a_3 + b_3}{2} \right)$$

we see that the midpoint of $A$ and $B$ is in $\mathbb{Z}^3$ if and only if $A$ and $B$ are of the same type. Since there are only 8 types and there are 9 points in $S$, there must be two points in $S$ with the same type, and therefore whose midpoint is in $\mathbb{Z}^3$.

**Remark.** This problem was once given in the Putnam competition.