Page 251, problem 7. Suppose \( m > 0 \) (the case \( m < 0 \) is similar.) Along the line \( y = mx \),
\[
  f(x, y) = f(x, mx) = \begin{cases} 
  0 & \text{if } (mx) \leq 0 \text{ or } mx \geq x^2, \\
  1 & \text{if } 0 < mx < x^2.
\end{cases}
\]
This means \( f(x, mx) = 0 \) unless \( x \) is positive and (dividing \( mx < x^2 \) by \( x \)) unless \( m < x \). In other words, for \( m \neq 0 \),
\[
  x \leq m \implies f(x, mx) = 0
\]
Hence \( \lim_{x \to 0} f(x, mx) = 0 \). Of course if \( m = 0 \), then \( f(x, mx) = f(x, 0) = 0 \), so the limit is also 0 in this case.

For the curve along which \( f = 1 \), we could use the parabola \( y = x^2/2 \).

The limit existed, it would have to be 0 (think of the line \( y = mx \)) and 1 (think of the parabola \( y = x^2/2 \)). This is impossible, so the limit does not exist.

Remark. Here is a formal proof that the limit does not exist. Suppose the limit did exist and was \( L \). Let \( \epsilon = 1/3 \). Then there is a \( \delta > 0 \) such that
\[
  (*) \quad 0 < |(x, y) - (0, 0)| < \delta \implies |f(x, y) - L| < 1/3.
\]
Letting \( y = 0 \) and \( 0 < x < \delta \) (for example \( x = \delta/2 \)), we see that
\[
  (**) \quad |f(x, 0) - L| < 1/3, \quad \text{or } |L| = |0 - L| < 1/3.
\]
Choosing \( x \) very close (but not equal) to 0 and letting \( y = x^2/2 \), we get that
\[
  0 < |(x, y) - (0, 0)| < \delta,
\]
so by (*),
\[
  (***) \quad |1 - L| = |f(x, y) - L| < 1/3.
\]
By (**) and (***) \( L \) has to be within 1/3 of both 0 and 1, which is impossible. Thus the limit does not exist. \( \square \)

Page 255, Problem 21(a). Proof that the ball \( \mathbb{B}(0, r) \) is convex: Let \( a \) and \( b \) be points in \( \mathbb{B}(0, r) \). Then \( |a| < r \) and \( |b| < r \). Let \( 0 \leq t \leq 1 \). Then
\[
  |(1 - t)a + tb| \leq |(1 - t)a| + |tb| \quad \text{(by the triangle inequality)}
\]
\[
  = (1 - t)|a| + t|b| \quad \text{(since } t \text{ and } 1 - t \text{ are } \geq 0)
\]
\[
  < (1 - t)r + tr
\]
\[
  = 1.
\]
Thus \((1-t)a + tb\) is in \(B(0,r)\). Thus \(B(0,r)\) is convex. \(\square\)

Proof that the ball \(B(x,r)\) is convex: Let \(a\) and \(b\) be points in \(B(x,r)\). Then \(|a - x| < r\) and \(|b - x| < r\). Let \(0 \leq t \leq 1\). Then

\[
\begin{align*}
|&(1-t)a + tb - x| = |(1-t)(a-x) + t(b-x)| \\
&\leq |(1-t)(a-x)| + |t(b-x)| \quad \text{(by the triangle inequality)} \\
&= (1-t)|a-x| + t|b-x| \quad \text{(since } t \text{ and } 1-t \text{ are } \geq 0) \\
&< (1-t)r + tr \\
&= 1.
\end{align*}
\]

Thus \((1-t)a + tb\) is in \(B(x,r)\). Thus \(B(x,r)\) is convex. \(\square\)

Remark. There is no need to do the \(B(0,r)\) case first. But I did it here because I thought that case would be easier for you to follow.

Page 256, problem 21(b). Let \(a\) and \(b\) be points in \(S\). Then \((1-t)a + tb\) is in \(S\) for \(0 \leq t \leq 1\). Let

\[
g(t) = f((1-t)a + tb) = f(a + t(b-a))
\]

Then

\[
g'(t) = f'(a + t(b-a); b-a) = 0.
\]

Since this is true for \(0 \leq t \leq 1\), we know (by one variable calculus) that \(g\) is constant on \([0,1]\). In particular, \(g(0) = g(1)\), which means \(f(a) = f(b)\). Since \(a\) and \(b\) are arbitrary points in \(S\), this implies that \(f\) is constant on \(S\). \(\square\)