MIDTERM REVIEW PROBLEMS

1. Solve each differential equation.
   (a) \( xy' - 2y - x^3 \sin x = 0 \)
   (b) \( y^3 - 4x + (3xy^2 + 1)y' = 0 \)
   (c) \( xy' - y + e^x y^2 = 0 \)
   (d) \( xy^2 y' = x^3 + y^3, y(1) = 2 \)
   (e) \( xy' = y(y - 2) \)
   (f) \( (x^2 + 1)y' = 2xy + 2x(x^2 + 1) \)
   (g) \( y' = \frac{3x - 2y}{y + 2x} \)
   (h) \( \frac{1}{x+1} x' + \frac{2}{x} \tan^{-1} x = \frac{2}{x} \)

2. Compute
   \[
   \det \begin{bmatrix}
   4 & 4^2 & 4^3 & 4^4 \\
   4^5 & 4^6 & 4^7 & 4^8 \\
   4^9 & 4^{10} & 4^{11} & 4^{12} \\
   4^{13} & 4^{14} & 4^{15} & 4^{16}
   \end{bmatrix}.
   \]

3. Let
   \[
   \begin{bmatrix}
   3 & -1 \\
   3 & -1
   \end{bmatrix}.
   \]
   (a) Determine the eigenvalues and eigenvectors of \( A \).
   (b) Compute \( A^{2005} \).

4. Let \( A \) be an \( n \times n \) matrix such that \( A^n \) is the zero matrix, and \( A^k \) is NOT the zero matrix for \( k = 1, 2, \ldots, n - 1 \). Let \( v \in \mathbb{R}^n \) such that \( v \) is NOT in the nullspace of \( A^{n-1} \). Prove that the set
   \[ \{ v, Av, A^2v, \ldots, A^{n-1}v \} \]
   is linearly independent.

5. If \( A \) is a square matrix, prove that \( A^T A = I \) if and only if the column vectors of \( A \) are mutually orthogonal unit vectors.
6. Let \( P_1 = (x_1, y_1) \) and \( P_2 = (x_2, y_2) \) be distinct points in \( \mathbb{R}^2 \). Show that the equation
\[
\det \begin{bmatrix}
x & y & 1 \\
x_1 & y_1 & 1 \\
x_2 & y_2 & 1
\end{bmatrix} = 0
\]
is that of a the straight line through \( P_1 \) and \( P_2 \).

7. Prove that taking the transpose of a matrix is a linear transformation of set of \( m \times n \) matrices to the set of \( n \times m \) matrices.

8. If \( A \) and \( B \) are matrices that commute, prove that \( A^2 \) and \( B^2 \) also commute.

9. A permutation matrix has one nonzero element in each row and one nonzero element in each column, and each nonzero element is equal to one. Show that the determinant of such a matrix is either 1 or -1.

10. Let \( A \) be a square matrix. Prove that \( A + A^T \) is a symmetric matrix.

11. If \( A \) is an \( n \times 1 \) matrix, and \( B \) is a \( 1 \times n \) matrix, with \( n > 1 \), prove that the \( n \times n \) matrix \( AB \) has determinant equal to zero.

12. Define the trace of an \( n \times n \) matrix to be the sum of the diagonal elements, that is \( tr(A) = a_{11} + a_{22} + \ldots + a_{nn} \).

   (a) Prove that \( tr(A) \) is a linear transformation from the set of \( n \times n \) matrices to \( \mathbb{R} \).

   (b) Prove that for all \( n \times n \) matrices \( A \) and \( B \), we have
   \[
   tr(AB) = tr(BA).
   \]

   (c) Prove that there do not exist square matrices such that
   \[
   AB - BA = I.
   \]