6.9. With \( P = 2x + y \) and \( Q = x - 6y \), we see that

\[
\frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x}
\]

so the equation is exact. We solve by setting

\[
F(x, y) = \int P(x, y) \, dx = \int (2x + y) \, dx = x^2 + xy + \phi(y).
\]

To find \( \phi \), we differentiate

\[
Q(x, y) = \frac{\partial F}{\partial y} = x + \phi'(y).
\]

Hence \( \phi' = -6y \), and we can take \( \phi(y) = -3y^2 \). Hence the solution is \( F(x, y) = x^2 + xy - 3y^2 = C \).

6.10. With \( P = 1 - y \sin x \) and \( Q = \cos x \), we see that

\[
\frac{\partial P}{\partial y} = -\sin x = \frac{\partial Q}{\partial x},
\]

so the equation is exact. We solve by setting

\[
F(x, y) = \int P(x, y) \, dx = \int (1 - y \sin x) \, dx = x + y \cos x + \phi(y).
\]

To find \( \phi \), we differentiate

\[
Q(x, y) = \frac{\partial F}{\partial y} = \cos x + \phi'(y).
\]

Thus \( \phi' = 0 \), so we can take \( \phi = 0 \). Hence the solution is \( F(x, y) = x + y \cos x = C \).

6.11. With \( P = 1 + \frac{y}{x} \) and \( Q = -\frac{1}{x} \), we compute

\[
\frac{\partial P}{\partial y} = \frac{1}{x} \neq \frac{\partial Q}{\partial x} = \frac{1}{x^2}.
\]

Hence the equation is not exact.

6.15. Exact \( u^2/2 + vu - v^2/2 = C \)

6.16. Exact \( \ln(u^2 + v^2) = C \)

6.17. Not exact

7.1. The right hand side of the equation is \( f(t, y) = 4 + y^2 \). \( f \) is continuous in the whole plane. Its partial derivative \( \partial f/\partial y = 2y \) is also continuous on the whole plane. Hence the hypotheses are satisfied and the theorem guarantees a unique solution.

7.2. The right hand side of the equation is \( f(t, y) = \sqrt[3]{y} \). \( f \) is defined only where \( y \geq 0 \), and it is continuous there. However, \( \partial f/\partial y = 1/(2\sqrt[3]{y}) \), which is only continuous for \( y > 0 \). Our initial condition is at \( y_0 = 0 \), and \( t_0 = 4 \). There is no rectangle containing \((0, y_0)\) where both \( f \) and \( \partial f/\partial y \) are defined and continuous. Consequently the hypotheses of the theorem are not satisfied.

7.3. The right hand side of the equation is \( f(t, y) = t \tan^{-1} y \), which is continuous in the whole plane. \( \partial f/\partial y = \frac{t}{1 + y^2} \) is also continuous in the whole plane. Hence the hypotheses are satisfied and the theorem guarantees a unique solution.

7.4. The right hand side of the equation is \( f(s, \omega) = \omega \sin \omega + s \), which is continuous in the whole plane. \( \partial f/\partial \omega = \sin \omega + \omega \cos \omega \) is also continuous in the whole plane. Hence the hypotheses are satisfied and the theorem guarantees a unique solution.

7.5. The right hand side of the equation is \( f(t, x) = t/(x + 1) \), which is continuous in the whole plane, except where \( x = -1 \). \( \partial f/\partial x = -t/(x + 1)^2 \) is also continuous in the whole plane, except where \( x = -1 \). Hence the hypotheses are satisfied in a rectangle containing the initial point \((0, 0)\), so the theorem guarantees a unique solution.
7.7. The equation is linear. The general solution is \( y(t) = t \sin t + Ct \). Several solutions are plotted in the following figure.

Since every solution satisfies \( y(0) = 0 \), there is no solution with \( y(0) = -3 \). If we put the equation into normal form

\[
\frac{dy}{dt} = t y + t \cos t,
\]

we see that the right hand side \( f(t, y) \) fails to be continuous at \( t = 0 \). Consequently the hypotheses of the existence theorem are not satisfied.

9.15. (i) In this case, \( f(y) = 2 - y \), whose graph is shown in the following figure.

(ii) The phase line is easily captured from the previous figure, and is shown in the following figure.

(iii) The phase line in the second figure indicates that solutions increase if \( y < 2 \) and decrease if \( y > 2 \). This allows us to easily construct the phase portrait shown in the \( t,y \) plane in the next figure. Note the stable equilibrium solution, \( y(t) = 2 \).
9.17. (i) In this case, \( f(y) = (y + 1)(y - 4) \), whose graph is shown in the next figure.

(ii) The phase line is easily captured from the previous figure, and is shown in the next figure.

(iii) The phase line in the second figure indicates that solutions increase if \( y < -1 \), decrease for \(-1 < y < 4 \), and increase if \( y > 4 \). This allows us to easily construct the phase portrait shown in the \( ty \) plane in the next figure. Note the unstable equilibrium solution, \( y(t) = 4 \), and the stable equilibrium solution, \( y(t) = -1 \).
9.19. (i) In this case, \( f(y) = 9y - y^3 \) factors as \( f(y) = y(y + 3)(y - 3) \), whose graph is shown in the next figure.

![Graph of f(y) = 9y - y^3](image)

(ii) The phase line is easily captured from the previous figure, and is shown in the next figure.

![Phase line graph](image)

(iii) The phase line in the second figure indicates that solutions increase if \( y < -3 \), decrease for \( -3 < y < 0 \), increase if \( 0 < y < 3 \), and decrease for \( y > 3 \). This allows us to easily construct the phase portrait shown in the \( ty \) plane in the next figure. Note the stable equilibrium solution, \( y(t) = -3 \), the unstable equilibrium solution, \( y(t) = 0 \), and the stable equilibrium solution, \( y(t) = 3 \).
9.20. (i) In this case, \( f(y) = (y + 1)(y^2 - 9) \) factors as \( f(y) = (y + 1)(y - 3)(y + 3) \), whose graph is shown in the next figure.

(ii) The phase line is easily captured from the previous figure, and is shown in the next figure.

(iii) The phase line in the second figure indicates that solutions decrease if \( y < -3 \), increase for \(-3 < y < -1 \), decrease if \(-1 < y < 3 \), and increase for \( y > 3 \). This allows us to easily construct the phase portrait shown in the \( ty \) plane in the next figure. Note the unstable equilibrium solution, \( y(t) = -3 \), the stable equilibrium solution, \( y(t) = -1 \), and the unstable equilibrium solution, \( y(t) = 3 \).

9.27. We have the equation \( x' = f(x) = 4 - x^2 \). The equilibrium points are at \( x = \pm 2 \), where \( f(x) = 0 \). We have \( f'(x) = -2x \). Since \( f'(-2) = 4 > 0 \), \( x = -2 \) is unstable. Since \( f'(2) = -4 < 0 \), \( x = 2 \) is asymptotically stable.

9.28. We have the equation \( x' = f(x) = x(x - 1)(x + 2) \). The equilibrium points are at \( x = 0, 1, \) and \(-2 \), where \( f(x) = 0 \). We have \( f'(x) = 3x^2 + 2x - 2 \). Since \( f'(0) = -\frac{2}{3} < 0 \), \( x = 0 \) is asymptotically stable. Because \( f'(1) = 3 > 0 \), \( x = 1 \) is unstable. Finally, because \( f'(-2) = 2 > 0 \), \( x = -2 \) is also unstable.

9.29. (a) \( f(x) = x^2 \), \( f(x) = x^3 \), or \( f(x) = x^4 \).

(b) \( f(x) = -x^2 \), \( f(x) = -x^3 \), or \( f(x) = -x^7 \).
1.9. If you multiply matrix $A$ by $[2, 0, 0]^T$, then

$$A \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = [a_1, a_2, a_3] \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = 2a_1.$$ 

You can use similar strategies to triple and quadruple the second and third columns, respectively. Thus,

$$\begin{pmatrix} -1 & 2 & 4 \\ 0 & 5 & 2 \\ -1 & -2 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} -2 & 6 & 16 \\ 0 & 15 & 8 \\ -2 & -6 & 12 \end{pmatrix}.$$ 

The columns of $AB$ are

$$Ab_1 = [a_1, a_2, a_3] \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = 2a_1,$$

$$Ab_2 = [a_1, a_2, a_3] \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} = 3a_2,$$

$$Ab_3 = [a_1, a_2, a_3] \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} = 4a_3.$$ 

Thus,

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$ 

1.31.

$$4v_1 - 3v_4 + 3v_2 + 4v_3 = 4 \begin{pmatrix} 10 \\ -5 \\ 3 \end{pmatrix} - 3 \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 8 \\ 6 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ -9 \\ 7 \end{pmatrix} = \begin{pmatrix} -20 \\ 0 \\ 18 \end{pmatrix} + \begin{pmatrix} -9 \\ 0 \\ -28 \end{pmatrix} = \begin{pmatrix} 43 \\ -18 \\ 28 \end{pmatrix}.$$ 

1.32.

$$v_2 - 5v_4 - 3v_2 + 2v_3 = \begin{pmatrix} 0 \\ 8 \\ 6 \end{pmatrix} - 5 \begin{pmatrix} 10 \\ 3 \\ 6 \end{pmatrix} - 3 \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ -9 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 6 \end{pmatrix} + \begin{pmatrix} -50 \\ 25 \\ -18 \end{pmatrix} = \begin{pmatrix} -47 \\ 42 \\ -14 \end{pmatrix}.$$ 

1.33.

$$Ay = \begin{pmatrix} 9 & -6 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 39 \\ 17 \end{pmatrix}.$$ 

1.34.

$$Ax_3 = \begin{pmatrix} 9 & -6 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 8 \end{pmatrix} = \begin{pmatrix} -57 \\ -13 \end{pmatrix}.$$ 

1.38.

$$Cw = \begin{pmatrix} 10 & 0 & -1 \\ -5 & 8 & 3 \\ 3 & 6 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 23 \\ -19 \\ -12 \end{pmatrix}.$$ 

1.39.

$$Cy_3 = \begin{pmatrix} 10 & 0 & -1 \\ -5 & 8 & 3 \\ 3 & 6 & 6 \end{pmatrix} \begin{pmatrix} 0 \\ -9 \\ 7 \end{pmatrix} = \begin{pmatrix} -7 \\ -51 \\ -12 \end{pmatrix}.$$ 

1.40.

$$Du = \begin{pmatrix} 10 & 0 & 0 \\ -5 & 8 & -9 \\ 3 & 6 & 7 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -14 \\ 59 \\ -25 \end{pmatrix}.$$
3.2. Solve for $x$.

$$-2x + 4y = 0$$
$$-2x = -4y$$
$$x = 2y$$

Thus, the solutions can be written

$$x = \begin{pmatrix} x \\ y \end{pmatrix} = \frac{2}{y} \begin{pmatrix} 2y \\ 1 \end{pmatrix},$$

where $y$ is free. Alternatively, you can let $y = \alpha$ and write $x = \alpha(2, 1)^T$, where $\alpha$ is any real number.

3.10. Set up the augmented matrix.

$$\begin{pmatrix} 18 & -12 & 96 \\ 8 & -2 & 36 \end{pmatrix}$$

Multiply the first row by $-8/18$ (or $-2/9$) and add to the second row.

$$\begin{pmatrix} 18 & -12 & 96 \\ 0 & 10/3 & -20/3 \end{pmatrix}$$

This gives

$$18x - 12y = 96,$$
$$\frac{10}{3}y = -\frac{20}{3}.$$ 

The second equation gives $y = -2$. Substitute this in the first equation.

$$18x - 12(-2) = 96$$
$$18x + 24 = 96$$
$$18x = 72$$
$$x = 4$$

The solution is $(4, -2)^T$.

3.14. In matrix form the system is

$$\begin{pmatrix} 10 & -5 & 3 \\ -20 & 18 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}.$$ 

The augmented matrix is

$$M = \begin{pmatrix} 10 & -5 & 3 & 2 \\ -20 & 18 & 0 & -2 \end{pmatrix}.$$ 

We reduce to row echelon form by adding 2 times the first row to the second:

$$M \rightarrow \begin{pmatrix} 10 & -5 & 3 & 2 \\ 0 & 8 & 6 & 2 \end{pmatrix}.$$ 

The simplified system is

$$10x - 5y + 3z = 2,$$
$$8y + 6z = 2,$$

To backsolve, we first set the free variable $z = t$. Then $y = (2-6t)/8 = (2-6t)/8$. Finally, $x = 2 + 5y - 3z = 2 + 5(2-6t)/8 - 3t = 13/4 - 27t/4$. Hence the solutions are the vectors

$$x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{13/4 - 27t/4}{1/4 - 3t/4} + t \begin{pmatrix} 13/4 \\ 1/4 \\ -27/4 \end{pmatrix}.$$ 

3.17. Set up the augmented matrix and reduce.

$$\begin{pmatrix} -6 & 8 & 0 & 2 \\ 4 & 8 & 8 & 20 \\ -2 & 2 & 7 & 7 \end{pmatrix}$$

This gives $x = (1, 1, 1)^T$. 

3.20. Set up the augmented matrix and reduce.

\[
\begin{pmatrix}
-4 & 10 & -6 & -14 \\
0 & -4 & 4 & 4 \\
2 & -10 & 8 & 12
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & -1 & -1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

This gives
\[
x - z = 1, \\
y - z = -1.
\]

Thus, \(x\) and \(y\) are pivot variables and \(z\) is free. Solve each equation for its pivot variable.

\[
x = 1 + z \\
y = -1 + z
\]

Thus,
\[
x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 + z \\ -1 + z \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},
\]

where \(z\) is free.

3.28. Set up the augmented matrix and reduce.

\[
\begin{pmatrix}
-7 & -4 & -5 & -9 & 31 \\
1 & 10 & 7 & 10 & -18 \\
-2 & 0 & -3 & 0 & -2
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 & 0 & 75/55 & -379/77 \\
0 & 1 & 0 & 19/14 & -57/14 \\
0 & 0 & 1 & -50/77 & 304/77
\end{pmatrix}
\]

This gives
\[
x_1 + \frac{75}{77}x_4 = -\frac{379}{77}, \\
x_2 + \frac{19}{14}x_4 = -\frac{57}{14}, \\
x_3 - \frac{50}{77}x_4 = \frac{304}{77}.
\]

Therefore, \(x_1, x_2,\) and \(x_3\) are pivot variables and \(x_4\) is free. Solve each equation for its pivot variable.

\[
x_1 = -\frac{379}{77} - \frac{75}{77}x_4 \\
x_2 = -\frac{57}{14} - \frac{19}{14}x_4 \\
x_3 = \frac{304}{77} + \frac{50}{77}x_4
\]

Thus,
\[
x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\frac{379}{77} - (75/77)x_4 \\ -\frac{57}{14} - (19/14)x_4 \\ 304/77 + (50/77)x_4 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -75/77 \\ -19/14 \\ 50/77 \\ 1 \end{pmatrix},
\]

where \(x_4\) is free.