Math 139  
Mathematics of Medical Imaging  
Problem Set # 1  
Due Thursday, January 20

The first set of problems concern the shadow function \( h_D(\theta) \) defined for a convex planar region \( D \). As in class, we parametrize the boundary curve as \((x(\theta), y(\theta))\), \(0 \leq \theta \leq 2\pi\).

1. Suppose \( D \) is translated by a vector \( v \in \mathbb{R}^2 \). Give an explicit description for the shadow function of this translated region in terms of the original shadow function \( h_D \). Similarly, describe the effect on the shadow function \( h_D \) of rotating \( D \) by an angle \( \theta_0 \).

2. Suppose that \( D_n \) is a regular convex polygon centered at the origin. What is its shadow function? How much can you say about the shadow function of a more general convex polygon?

3. Let \( h(\theta) \) be any twice differentiable, \(2\pi\)-periodic function which satisfies \( h''(\theta) + h(\theta) > 0 \) for every \( \theta \). Define a curve \( \gamma \) which is parametrized by 
\[
(x(\theta), y(\theta)) = h(\theta)\omega(\theta) + h'(\theta)\dot{\omega}(\theta),
\]
where \( \omega(\theta) = (\cos \theta, \sin \theta) \) and \( \dot{\omega}(\theta) = \omega'(\theta) \). Show that \( \gamma \) is the boundary of a convex region. Prove that the curvature of \( \gamma \) is equal to \( 1/(h''(\theta) + h(\theta)) \). (Recall that the curvature \( \kappa = |dN/ds| \), where \( N \) is the unit normal vector to \( \gamma \) and \( s \) is the arclength parameter along \( \gamma \).)

4. Let \( A \) be any invertible \( n \times n \) matrix. Show that its condition number \( c_A \) is also given by the expression
\[
c_A = \left( \max_{x \neq 0} \frac{||Ax||}{||x||} \right) / \left( \min_{x \neq 0} \frac{||Ax||}{||x||} \right).
\]

5. Suppose that \( A \) is a symmetric matrix and that its eigenvalues are \( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \). Show that \( c_A = \lambda_n / \lambda_1 \). For a general (not necessarily symmetric) matrix \( A \), form the symmetric matrix \( A^*A \). Show that 
\[
c_A = \sqrt{c_A A^*A}.
\]