Math 174b, Problem Set 3

Due Friday, 12 May

1. Suppose that $u$ solves the inhomogeneous wave equation $\partial_t^2 u = \triangle u + F(x, t)$. Let $u'$ denote the space and time derivatives of $u$: $(\partial_t u, \nabla_x u)$. Recall that the energy of the function $u$ at time $t$ is denoted $E(t)$ and is given by the formula $E(t) = \int_{\mathbb{R}^n} |u'|^2 dx$. Prove the following formula for the time-derivative of the energy.

$$\frac{d}{dt} E(t) = 2 \int_{\mathbb{R}^n} F(x, t) \partial_t u(x, t) dx.$$ 

You may recall from high school physics that Energy = Force $\times$ distance. Therefore, $\frac{d}{dt}$ Energy = Force $\times$ velocity. This high-school formula is consistent with the formula above, because $\partial_t u$ can be thought of as a velocity (say of an elastic membrane).

Using the last formula and the Cauchy-Schwarz inequality, give another proof of the following inequality.

$$E(t)^{1/2} \leq E(0)^{1/2} + \int_0^t \|F(\cdot, \tau)\|_2 d\tau.$$ 

2. Let $p$ be a number in the range $0 < p < 1$. Let $f$ be a Schwartz function on $\mathbb{R}^n$. Suppose that $f$ obeys the following inequality:

$$\int_{\mathbb{R}^n} |\nabla f|^p dx \leq 1.$$ 

Prove that nevertheless the function $f$ may be arbitrarily large. More precisely, given any radius $R$ and any number $M$, it may happen that $f(x) \geq M$ for $x$ in the ball of radius $R$ around 0.

Conclude that if $p < 1$, then there is no analogue of the Sobolev inequality involving $\|\nabla f\|_p$.

3. Let $u(x, t)$ be a function on $\mathbb{R}^3 \times \mathbb{R}$, solving the wave equation $\partial_t^2 u = \triangle u$, with initial conditions $u(x, 0) = 0$ and $\partial_t u(x, 0) = g(x)$. Recall that $u(x, t)$ is given by the explicit formula $u(x, t) = t M_t g(x)$, where $M_t g(x)$ denotes the average value of $g$ on the sphere of radius $t$ around $x$. Prove the following estimate controlling the decay of the function $u$, for $t > 0$.

$$|u(x, t)| \leq C (1 + t)^{-1} (\|g\|_1 + \|\nabla g\|_1 + \|\nabla^2 g\|_1).$$ 

[Hint: Try the cases $t > 1$ and $t < 1$ separately. I think $t > 1$ is easier.]

4. (open-ended problem) Using the slicing idea, return to problem 4 on the last homework. If you solve it, here are some variations you could tackle.

a. Let $U$ be a bounded open set in $\mathbb{R}^n$. Let $U_i$ denote the projection of $U$ onto the hyperplane perpendicular to the $x_i$ axis. Let $|U|$ denote the volume of $U$ and $|U_i|$ denote the (n-1)-dimensional volume of $U_i$. Prove the following formula.

$$|U| \leq \prod_{i=1}^n |U_i|^\frac{1}{n-1}.$$ 

b. Let $f$ be a Schwartz function on $\mathbb{R}^n$. Let $f_i$ be the function defined on the coordinate plane perpendicular to the $x_i$ axis by the following formula.
\( f_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = \sup_{t \in \mathbb{R}} |f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)|. \)

Prove the following function-theoretic analogue of the inequality in a.

\[
\|f\|_{\infty} \leq \prod_{i=1}^{n} \|f_i\|_{1}^{1/n}.
\]

c. Using the fundamental theorem of calculus, prove that \( \|f_i\|_{1} \leq (1/2)\|\partial_i f\|_{1} \).

Combining this result with the one in b., prove the Gagliardo-Nirenberg-Sobolev inequality.

\[
\|f\|_{\infty} \leq \prod_{i=1}^{n} \|\partial_i f\|_{1}^{1/n}.
\]

d. Let \( U \) be a bounded open set in \( \mathbb{R}^n \). Let \( P_I \) be the \( k \)-dimensional coordinate plane with axes \( x_{i_1}, \ldots, x_{i_k} \), where \( I \) is the sequence of indices \((i_1, \ldots, i_k)\). Let \( U_I \) denote the projection of \( U \) onto \( P_I \). In the notation of part a., \( U_1 \) corresponds to \( U_I \) for \( I = (2, \ldots, n) \). Let \( |U| \) denote the volume of \( U \), and let \( |U_I| \) denote the volume of \( U_I \). Figure out what happens.