Math 174b, Problem Set 1

Due Friday, 14 April

1. Suppose \( \|f\|_1 = A \) and \( \|f\|_\infty = B \). Prove that \( \|f\|_2 \leq \sqrt{AB} \). More generally, prove that \( \|f\|_p \leq A^{\frac{1}{p}} B^{1-\frac{1}{p}} \).

2. Suppose that \( f \) is supported on the interval \([0, L]\) of length \( L \). Prove the following inequality called Bernstein’s inequality for any \( 1 \leq p < q \).

\[
\|f\|_p \leq L^{\frac{1}{p} - \frac{1}{q}} \|f\|_q.
\]

[Hint: Use Holder’s inequality.]

3. Suppose that \( \Psi \) is a function on the real line with \( \|\Psi\|_2 = 1 \). For instance, \( \Psi \) could be the state function of a particle in quantum mechanics. Prove the following inequality in the spirit of the Heisenberg uncertainty principle.

\[
\|\Psi\|_1 \|\hat{\Psi}\|_1 \geq 1.
\]

4. Suppose that \( f \) is a function on the real line with \( \|f\|_2 = 1 \). Prove that there are functions \( g \) and \( h \) so that \( f = g + h \) and \( \|g\|_1 + \|h\|_\infty < 100 \).

5. (Open-ended problem) For any function \( f \) on the real line, define \( Tf \) to be the function

\[
Tf(x) = \int_{x-1}^{x+1} f(y)dy.
\]

Try to find all possible inequalities of the form \( \|Tf\|_q < C\|f\|_p \).