Math 174a, Problem Set 4

Due Friday, 10 February

1. Suppose that $f$ is a $C^1$ function on the circle and that the average value of $f$ is zero, in other words $\frac{1}{2\pi} \int_0^{2\pi} f(x) dx = 0$. Prove the following inequality.

$$\int_0^{2\pi} |f(x)|^2 dx \leq \int_0^{2\pi} |f'(x)|^2 dx.$$  
I believe that this is called Wirtinger’s inequality.

The following questions concern the Fourier transform. We will cover it starting on Monday, following chapter 5 of Stein and Shakarchi. Exercises 3 and 5 are pretty hard. By Monday I will e-mail you some hints. There are also hints in the textbook. Exercise 3 is exercise 3 from chapter 5, and exercise 5 is exercise 5 from chapter 5.

2. Let $f(x)$ be equal to 1 if $|x| \leq 1$ and 0 otherwise. Calculate the Fourier transform of $f$ and check that $\hat{f}(\xi) = \frac{\sin 2\pi \xi}{\pi \xi}$.

3. Suppose that $f$ is a function of moderate decrease on $\mathbb{R}$ whose Fourier transform $\hat{f}$ is continuous and satisfies

$$\hat{f}(\xi) = O\left(\frac{1}{1 + |\xi|^{1+\alpha}}\right),$$
for some $0 < \alpha < 1$. Using the Fourier inversion formula, prove that $f$ is $C^\alpha$. In other words, prove that for some number $M$, the following inequality holds for any real numbers $x$ and $h$:

$$|f(x + h) - f(x)| < M|h|^\alpha.$$  

4. We say that a function has compact support if it is zero outside of some finite interval. Examples of compactly supported Schwartz functions are useful in many situations.

Suppose $a < b$. Define a function $f(x)$ that vanishes when $x \leq a$ or when $x \geq b$, and which is defined by the formula

$$f(x) = e^{-1/(x-a)} e^{-1/(x-b)}$$
when $a < x < b$. Show that the $k$th derivative of $f$ is well-defined and continuous for every positive integer $k$. In other words, show that $f$ is $C^k$ for every $k$.

5. Suppose that $f$ is a continuous function of moderate decrease on $\mathbb{R}$. Prove that $\hat{f}$ is also continuous and that $\hat{f}(\xi) \rightarrow 0$ as $|\xi|$ goes to infinity.