Math 42 Autumn 2004 Homework 5 Solutions

6.7 #2 (a) \( \int_{0}^{15} f(t) dt \) (b) \( \int_{-\infty}^{\infty} f(t) dt \)

6.7 #4 (a) \( f(x) \geq 0 \) for all \( x \), and \( \int_{-\infty}^{\infty} f(x) dx = \int_{0}^{10} f(x) dx = \frac{1}{2}(0)(0.2) = 1 \), using the formula for the area of a triangle.

(b) (i) \( P(X < 3) = \int_{0}^{3} f(x) dx = \frac{1}{3}(0.1) = 0.15 \). (ii) \( P(X > 8) = \int_{8}^{10} f(x) dx = \frac{1}{2}(0.1) = 0.1 \). So \( P(3 \leq X \leq 8) = 1 - P(X < 3) - P(X > 8) = 0.75 \).

(c) We find equations of the lines from (0, 0) to (6, 0.2) and from (6, 0.2) to (10, 0), and find that

\[
 f(x) = \begin{cases} 
 \frac{\pi}{16} x & \text{if } 0 \leq x < 6, \\
 -\frac{1}{20} x + \frac{1}{2} & \text{if } 6 \leq x \leq 10, \\
 0 & \text{otherwise.}
\end{cases}
\]

So the mean is

\[
 \mu = \int_{-\infty}^{\infty} x f(x) dx = \int_{0}^{6} x \left( \frac{1}{30} x \right) dx + \int_{6}^{10} x \left( -\frac{1}{20} x + \frac{1}{2} \right) dx = \frac{16}{3}.
\]

6.7 #6 (a) \( \mu = 1000 \) so

\[
 f(x) = \begin{cases} 
 0 & \text{if } t < 0, \\
 \frac{1}{1000} e^{-t/1000} & \text{if } t \geq 0.
\end{cases}
\]

(i) \( P(0 \leq X \leq 200) = \int_{0}^{200} \frac{1}{1000} e^{-t/1000} dt = -e^{-1/5} + 1 \approx 0.181 \).

(ii) \( P(X > 800) = \int_{800}^{\infty} \frac{1}{1000} e^{-t/1000} dt = \lim_{x \to \infty} \left[ -e^{-t/1000} \right]_{800} = e^{-8} \approx 0.449 \).

(b) We need to find \( m \) so that \( \int_{m}^{\infty} f(t) dt = \frac{1}{2} \). Since

\[
 \int_{m}^{\infty} \frac{1}{1000} e^{-t/1000} dt = \lim_{x \to \infty} \left[ -e^{-t/1000} \right]_{m} = e^{-m/1000},
\]

we solve \( e^{-m/1000} = \frac{1}{2} \) and get \( m = 1000 \ln 2 \approx 693 \) h.

6.7 #8 (a) With \( \mu = 69 \) and \( \sigma = 2.8 \), we have

\[
 P(65 \leq X \leq 73) = \int_{65}^{73} \frac{1}{(2.8)\sqrt{2\pi}} \exp \left( -\frac{(x - 69)^2}{2(2.8)^2} \right) dx \approx 0.847
\]

(using a calculator or computer for the numerical estimate).

(b) \( P(X > 6 \text{ feet}) = P(X > 72 \text{ inches}) = 1 - P(0 \leq X \leq 72) = 1 - 0.858 = 0.142 \) (again using a computer for the numerical estimate of the definite integral expression), so about 14.2% of the adult male population is more than 6 feet tall.

7.1 #2 \( y' = \cos^2 x - \sin^2 x + \sin x, \) so

\[
 y' + (\tan x)y = \cos^2 x - \sin^2 x + \sin x + \sin^2 x - \sin x = \cos^2 x.
\]

Also, \( y(0) = \sin 0 \cos 0 - \cos 0 = -1 \), so the initial condition is satisfied.

7.1 #4 If \( y = e^{rt} \), then \( y' = re^{rt} \) and \( y'' = r^2 e^{rt} \). So \( y'' + y' - 6y = 0 \) if and only if \( e^{rt}(r^2 + r - 6) = 0 \). Since \( e^{rt} \neq 0 \), this means we must have \( r^2 + r - 6 = (r + 3)(r - 2) = 0 \), so \( r = -3 \) or 2.

7.1 #8 (a) If \( x \) is close to 0, then \( xy^3 \) is close to 0, so \( y' \) is close to 0 and the graph of \( y \) has a tangent line which is nearly horizontal. If \( x \) is large, then \( xy^3 \) is large (or far in the negative direction), and so the graph of \( y \) has a tangent line which is nearly vertical.

(b) \( y' = -\frac{1}{2}(e - x^2)^{-3/2}(-2x) = xe - x^3 = xy^3. \)

(d) Try a solution of the form in part (b) and solve for \( c = y(0) = e^{-1/2} \), so \( c = \frac{1}{4} \), and \( y = (4 - x^2)^{-1/2}. \)
7.1 #12 We consider each equation in turn.

A. \( y' = 1 + xy > 1 \) for points in the first quadrant, but we can see that \( y' < 0 \) for some points in the first quadrant.

B. \( y' = -2xy = 0 \) when \( x = 0 \), but we can see that \( y' > 0 \) for \( x = 0 \).

Therefore A and B are incorrect, so the correct equation is C.

7.1 #14 (a) The coffee cools most quickly as soon as it is removed from the heat source. The rate of cooling decreases toward 0 since the coffee approaches room temperature.

(b) We assume that the temperature of the room is constant (the room is too big to be warmed up noticeably by the coffee).

\[
\frac{dy}{dt} = -k(y - 20),
\]

where \( y \) is the temperature of the coffee, 20 is the room temperature, and \( k \) is a proportionality constant. The minus sign as been included so that the constant \( k \) is positive, since \( y \) is decreasing when \( y > 20 \). The initial condition is \( y(0) = 95 \). The answer to (a) and the model support each other, because as \( y \) approaches 20, \( y' \) approaches 0.

7.2 #10 (a) If \( y = c \), then \( y' = 0 \), so \( 0 = c^4 - 6c^3 + 5y^2 = c^2(c - 1)(c - 5) \), so \( c = 0, 1, \) or \( 5 \).

(b) \( y \) is increasing if \( y'(y - 1)(y - 5) > 0 \), which is true if \( y \) is in \( (-\infty, 0) \cup (0, 1) \cup (5, \infty) \).

(c) \( y \) is decreasing if \( y'(y - 1)(y - 5) < 0 \), which is true if \( 1 < y < 5 \).

7.2 #18 \( y = \pm 1, \pm 2 \) are equilibrium solutions. The slope is positive for \( y > 2, -1 < y < 1 \), and \( y < -2 \); and the slope is negative for \(-2 < y < -1 \) and \( 1 < y < 2 \). For the limiting behavior of solutions:

- If \( y(0) < -2 \), then \( \lim_{t \to -\infty} y = -2 \) and \( \lim_{t \to -\infty} = -\infty \).
- If \(-2 < y(0) < -1 \), then \( \lim_{t \to -\infty} y = -2 \) and \( \lim_{t \to -\infty} = -1 \).
- If \(-1 < y(0) < 1 \), then \( \lim_{t \to -\infty} y = 1 \) and \( \lim_{t \to -\infty} = -1 \).
- If \( 1 < y(0) < 2 \), then \( \lim_{t \to -\infty} y = 1 \) and \( \lim_{t \to -\infty} = 2 \).
- If \( y(0) > 2 \), then \( \lim_{t \to -\infty} y = \infty \) and \( \lim_{t \to -\infty} = 2 \).

7.2 #20 As \( x \) increases, the slopes decrease and all of the estimates are above the true values. Thus, all of the estimates are overestimates.

7.2 #22 \( h = 0.1 \), \( x_0 = 0 \), \( y_0 = 0 \), and \( F(x, y) = 1 - xy \). Then \( x_i = x_0 + hi = (0.1)i \), and \( y_{i+1} = y_i + hF(x_i, y_i) = y_i + (0.1)(1 - (0.1)i y_i) \). We obtain

\[
y_1 = 0.2, \ y_2 = 0.392, \ y_3 = 0.56064, \ y_4 = 0.6933632, \ y_5 = 0.782425088.
\]

Thus \( y(1) \approx 0.7824 \).

7.2 #28 (a) From Exercise 7.1.14, we have \( y' = -k(y - 20) \). We are given that \( y' = -1 \) when \( y = 70 \), so we solve \( -1 = -k(70 - 20) \) to get \( k = \frac{1}{50} \). So the differential equation becomes \( y' = -\frac{1}{50}(y - 20) \).

(b) The limiting value of the temperature is 20° C; that is, the temperature of the room.

(c) We have \( h = 2 \), \( t_0 = 0 \), \( y_0 = 95 \), \( F(t, y) = -\frac{1}{50}(y - 20) \). So \( t_i = 2i \) and \( y_{i+1} = y_i - \frac{1}{50}(y_i - 20) \). We get

\[
y_1 = 92, \ y_2 = 89.12, \ y_3 = 86.3552, \ y_4 = 83.700992, \ y_5 = 81.15295232.
\]

So \( y(10) \approx 81.15^\circ C \).